FACTORIZATION OF COMPLETE GRAPHS INTO THREE FACTORS WITH THE SMALLEST DIAMETER EQUAL TO 3 OR 4

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ABSTRACT. We determine the set of all positive integers $n$ for which the complete graph of order $n$ decomposes into three factors of given finite diameters $d_1 \leq d_2 \leq d_3$ in the case when $d_1$ is equal to 3 or 4.

1. INTRODUCTION

A factor of a graph $G$ is a subgraph of $G$ containing all the vertices of $G$. We say that $G$ has a factorization (a decomposition into factors) if there exists a collection $F_1, F_2, ..., F_m$ of factors such that each edge of $G$ belongs to exactly one of the factors. Graph factorizations have been extensively studied for many years. The study of decompositions of complete graphs into factors with given diameters was initiated by Bosák, Rosa and Znám [2] who proved the following fundamental result: If the complete graph $K_n$ is decomposable into $m$ factors $F_1, F_2, ..., F_m$ with prescribed diameters $d_1, d_2, ..., d_m$, then for every $n' > n$ the complete graph $K_{n'}$ is also decomposable into such factors. Therefore there exists the smallest positive integer $F(d_1, d_2, ..., d_m)$ such that for any $n \geq F(d_1, d_2, ..., d_m)$ the complete graph $K_n$ admits a decomposition into $m$ factors with prescribed diameters $d_1, d_2, ..., d_m$. For information about the rich literature on the subject we recommend [1-3].

In what follows we focus on the special case of decomposition into three factors. Let us assume that the diameters $d_1, d_2, d_3$ satisfy $d_1 \leq d_2 \leq d_3 < \infty$. A construction of Bosák, Rosa and Znám [2] of decomposition of $K_{d_1+d_2+d_3-8}$ into three factors with the smallest diameter $d_1 \geq 5$ yields

$$F(d_1, d_2, d_3) \leq d_1 + d_2 + d_3 - 8.$$ 

A substantial progress was achieved by Hrnčiar [4] who proved that for $d_1 > 65$ the graph $K_{d_1+d_2+d_3-9}$ is not decomposable into three factors, which
gives
\[ F(d_1, d_2, d_3) = d_1 + d_2 + d_3 - 8. \]

Bosák, Rosa and Znám also began investigating decompositions of complete graphs into three factors with the smallest diameter \( d_1 = 2 \). In [2] they derived a solution of this problem except for the value of \( F(2, 2, 2) \) for which they proved the inequality \( 12 \leq F(2, 2, 2) \leq 13 \). The exact value was found by Stacho and Urland with the help of a computer search. In [7] they proved that \( K_{12} \) cannot be decomposed into three factors with diameters equal to 2, and therefore \( F(2, 2, 2) = 13 \).

In this paper we present a complete solution to the problem of decomposing a complete graph into three factors with the smallest diameter equal to 3 or 4.

2. AN AUXILIARY RESULT

We state an auxiliary result about the number of common neighbours of a pair of vertices under certain hypotheses. In the proof we will be using the following Lemma A, proved in [2].

**Lemma A** Let \( G \) be a graph with \( n \) vertices and diameter \( d \) (\( 1 < d < n \)). Then the degree of every vertex in \( G \) is at most \( n - d + 1 \).

**Lemma 1** Let \( u, v \) be two distinct vertices of a graph \( G \) with \( n \) vertices (\( n \geq 5 \)) and diameter \( d \) (\( 2 \leq d \leq n - 3 \)). Let \( \deg(u) = a, \deg(v) = b \), where \( a, b \leq n - d + 1 - t \) and \( a + b > n - d + 3 \). Let \( A_1 \) be a fixed diameter path of \( G \). Suppose that \( y_1, y_2, \ldots, y_t, 0 \leq t \leq n - d - 3 \), are the vertices not in \( A_1 \) adjacent to neither \( u \) nor \( v \) in \( G \).

I. If \( uv \in E(G) \), then there exist at least \( (a + b) - (n - d + 3) + t \) vertices adjacent to both \( u \) and \( v \) in \( G \).

II. If \( uv \notin E(G) \), then there exist at least \( (a + b) - (n - d + 2) + t \) vertices adjacent to both \( u \) and \( v \) in \( G \).

**Proof** Let \( a + b = n - d + 4 + k \) where \( k \) is a nonnegative integer, and let \( a, b \leq n - d + 1 - t \). The set of all vertices adjacent to the vertex \( u \) (\( v \)) will be denoted by \( A \) (\( B \)). Note that the number of vertices in the fixed diameter path \( A_1 \) is \( d + 1 \) and the number of vertices not in \( A_1 \) is \( n - d - 1 \). If an arbitrary vertex not in \( A_1 \) is adjacent to three vertices in \( A_1 \), then one of these vertices in \( A_1 \) must be adjacent to both the remaining vertices in \( A_1 \).
Suppose that \( y_1, y_2, \ldots, y_t, \ 0 \leq t \leq n - d - 3 \), are the vertices not in \( A_1 \) adjacent neither to \( u \) nor to \( v \) in \( G \).

I. Let \( uv \in E(G) \). We first prove that \( |A \cup B| \leq (n - d + 3) - t \). We consider three cases:

a) Let \( u, v \in A_1 \). Both \( u \) and \( v \) can be adjacent to at most two vertices in \( A_1 \) and the same \((n - d - 1) - t \) vertices not in \( A_1 \). Thus \( |A \cup B| \leq 2 + 2 + (n - d - 1) - t = (n - d + 3) - t \).

b) Let \( u \in A_1, \ v \notin A_1 \). Except for \( u \), the vertex \( v \) is adjacent to at most two vertices \( v_1, v_2 \in A_1 \) (possibly \( v_1 = v_2 \)). The vertex \( u \) is adjacent to at most two vertices \( u_1, u_2 \in A_1 \). If \( v_1 \neq v_2 \) (as \( uv \in E(G) \)), at least one of these vertices coincides with some vertex \( u_i, i \in \{1, 2\} \). Therefore there exist at most four vertices in \( A_1 \) and \((n - d - 1) - t \) vertices not in \( A_1 \) contained in \( A \cup B \). It follows that \( |A \cup B| \leq (n - d + 3) - t \).

c) Let \( u \notin A_1, \ v \notin A_1 \). The vertex \( u \) (or \( v \)) is adjacent to at most three vertices \( u_i, 1 \leq i \leq 3 \) (or \( v_j, 1 \leq j \leq 3 \)), in \( A_1 \). Again, we prove that there exist at most four distinct vertices in \( A_1 \) contained in \( A \cup B \). If there exist at least five such vertices in \( A_1 \), then there would exist vertices \( u_i, v_j \), where \( i, j \in \{1, 2, 3\} \) such that in \( A_1 \) we have \( \rho(u_i, v_j) = 4 \). On the other hand, we have the path \( u_iuvv_j \) of length 3 in \( G \), which contradicts the equality \( \rho(u_i, v_j) = 4 \).

From the facts that \( a + b = |A| + |B| = n - d + 4 + k \) and \( |A \cup B| \leq (n - d + 3) - t \) it follows that \( |A \cap B| = |A| + |B| - |A \cup B| \geq (n - d + 4 + k) - [(n - d + 3) - t] = (a + b) - (n - d + 3) + t \).

II. Let \( uv \notin E(G) \). We first prove that \( |A \cup B| \leq (n - d + 2) - t \). Again, we distinguish three cases:

a) Let \( u, v \in A_1 \). Denote by \( u_i, 1 \leq i \leq 2 \) (or \( v_j, 1 \leq j \leq 2 \)), the vertices in \( A_1 \) adjacent to \( u \) (or \( v \)). Since \( a + b = n - d + 4 + k \), there are at least \( n - d \) edges in \( G \) joining \( u \) or \( v \) with the vertices not in \( A_1 \). Because the number of vertices not in \( A_1 \) adjacent to \( u \) or \( v \) is at least \((n - d - 1) - t \), obviously there exists at least one vertex \( i \notin A_1 \) adjacent to both \( u \) and \( v \) in \( G \). Then we have \( \rho(u, v) = 2 \) in \( A_1 \) and hence there exists an \( i \in \{1, 2\} \) and a \( j \in \{1, 2\} \) such that \( u_i = v_j \). Therefore \( |A \cup B| \leq 3 + (n - d - 1) - t = (n - d + 2) - t \).

b) Let \( u \in A_1, \ v \notin A_1 \). The vertex \( u \) is adjacent to at most two vertices \( u_i \in A_1 \ (1 \leq i \leq 2) \) and the vertex \( v \) is adjacent to at most three vertices \( v_j \in A_1 \ (1 \leq j \leq 3) \). Since \( a + b = n - d + 4 + k \), the graph \( G \) contains at least \( n - d - 1 \) edges joining \( u \) or \( v \) with the vertices not in \( A_1 \). Except for the vertex \( v \), there are \( n - d - 2 \) vertices not in \( A_1 \), therefore there exists at least one vertex \( x \notin A_1 \) adjacent to both \( u \) and \( v \) in \( G \). Suppose that the
vertices $u_1, u_2, v_1, v_2, v_3$ are distinct. Then $\rho(u, v_j) \geq 2$ for all $j \in \{1, 2, 3\}$ and hence there exists the vertex $v_j$ such that $\rho(u, v_j) = 4$. At the same time, we have the path $uxv_jv$ of length 3 in $G$, a contradiction. Consequently, $|A \cup B| \leq 4 + (n - d - 2) - t = (n - d + 2) - t$.

c) Let $u \notin A_1, v \notin A_1$. The vertex $u$ (or $v$) is adjacent to at most three vertices $u_i, 1 \leq i \leq 3$ ($v_j, 1 \leq j \leq 3$), from $A_1$. Since $a + b = n - d + 4 + k$, there are at least $n - d - 2$ edges in $G$ joining $u$ or $v$ with the vertices not in $A_1$. Except for $u$ and $v$, the number of the vertices not in $A_1$ is $n - d - 3$ and analogously as in II b) it can be easily shown that it is not possible to have six distinct vertices $u_1, u_2, v_1, v_2, v_3$ in $A_1$. It follows that $|A \cup B| \leq 5 + (n - d - 3) - t = (n - d + 2) - t$. Since $a + b = |A| + |B| = n - d + 4 + k$ and $|A \cup B| \leq (n - d + 2) - t$, we get $|A \cap B| = |A| + |B| - |A \cup B| \geq (n - d + 4 + k) - [(n - d + 2) - t] = (a + b) - (n - d + 2) + t$. This concludes the proof. 

We note that Lemma 1 has been presented in a more general form than actually needed. In particular, in most cases we will use it in situations where we do not assume the existence of some vertices adjacent neither to $u$ nor to $v$ in $G$.

3. RESULTS: THE CASE $d_1 = 3$

For $d_2 \leq 8$ and any $d_3 \geq d_2$ the exact values of the function $F(3, d_2, d_3)$ were presented by Palumbíny [5]. Further, for $d_2 \geq 9$ he proved that

$$F(3, d_2, d_3) \leq d_2 + d_3 - 6.$$  \hspace{1cm} (1)

Motivated by this result, we started with attempts to determine the values of $F(3, d_2, d_3)$ for any $d_2$ and $d_3$. Our investigation was successful, because we show that the complete graph of order $d_2 + d_3 - 7$ is not decomposable into three factors with diameters 3, $d_2$ and $d_3$.

**Theorem 1** Let $d_2 \geq 9$. Then $F(3, d_2, d_3) = d_2 + d_3 - 6$.

A part of the following proof was published in a lesser known journal [8]. To make this paper self-contained, we give a full proof here.

**Proof** By (1), $F(3, d_2, d_3) \leq d_2 + d_3 - 6$. We have to show that the graph $K_{d_2+d_3-7}$ is not decomposable into three factors $F_1, F_2, F_3$, with diameters 3, $d_2$ and $d_3$. Let us prove this by contradiction and assume that such a
decomposition exists. Denote by $A_1 = u_1pqu_2$ a fixed diameter path of the factor $F_1$. Invoking Lemma A it follows that the degree of every vertex in $F_2$ ($F_3$) is at most $(d_2 + d_3 - 7) - d_2 + 1 = d_3 - 6$ ($(d_2 + d_3 - 7) - d_3 + 1 = d_2 - 6$), thus every vertex of $F_1$ has degree greater than or equal to $(d_2 + d_3 - 8) - (d_2 - 6) - (d_3 - 6) = 4$. Let us denote the degree of the vertex $u_1$ ($u_2$) in $F_1$ by $\deg(u_1) = a$ $(\deg(u_2) = b)$. Here $a \geq 4$, $b \geq 4$ and $a + b \leq d_2 + d_3 - 9$, because the number of vertices not in $A_1$ is $d_2 + d_3 - 11$. Since there are no vertices adjacent to both $u_1$ and $u_2$ in $F_1$, we have $(d_2 + d_3 - 9) - (a + b)$ vertices adjacent neither to $u_1$ nor to $u_2$ in $F_1$.

Without lost of generality we may suppose that the edge $u_1u_2 \in E(F_2)$. We denote the degree of $u_i$, $i = 1, 2$, in $F_2$ by $\deg(u_1) = c$ and $\deg(u_2) = d$. The degree of any vertex in $F_3$ is at most $d_2 - 6$ and the degree of $u_1$ ($u_2$) in $F_1$ is $a$ ($b$), hence in $F_2$ we have $(d_2 + d_3 - 8) - (d_2 - 6) - a = d_3 - 2 - a \leq c \leq d_3 - 6$ ($(d_2 + d_3 - 8) - (d_2 - 6) - b = d_3 - 2 - b \leq d \leq d_3 - 6$). By part I of Lemma 1, if $c + d > d_3 - 4 = (d_2 + d_3 - 7) - d_2 + 3$ $(c + d \leq d_3 - 4)$, then there exist at least $c + d - (d_3 - 4)$ vertices (alternatively no vertex at all) adjacent to both $u_1$ and $u_2$ in $F_2$.

Observe that the number $c + d - (d_3 - 4)$ of vertices adjacent to both $u_1$ and $u_2$ in $F_3$ must be less than or equal to the number $d_2 + d_3 - 9 - (a + b)$ of vertices adjacent neither to $u_1$ nor to $u_2$ in $F_1$. Let us denote the quantity $d_2 + d_3 - 9 - (a + b) - (c + d - d_3 + 4) = d_2 + 2d_3 - 13 - a - b - c - d$ by $k$, clearly $k \geq 0$. We show that the number of vertices adjacent to both $u_1$ and $u_2$ in $F_3$ exceeds $k$, which will give a contradiction. In $F_3$, $\deg(u_1) = d_2 + d_3 - 8 - a - c$ and $\deg(u_2) = d_2 + d_3 - 8 - b - d$. Note that in $F_3$ we can not have $\deg(u_1) + \deg(u_2) \leq d_2 - 4$. Indeed, in the opposite case we would get $2(d_2 + d_3 - 8) - a - b - c - d \leq d_2 - 4$ and consequently, $d_2 + 2d_3 - 12 - a - b - c - d = k + 1 \leq 0$.

It follows that $\deg(u_1) + \deg(u_2) > d_2 - 4 = (d_2 + d_3 - 7) - d_3 + 3$ and by part II of Lemma 1, the number of vertices adjacent to both $u_1$ and $u_2$ in $F_3$ is at least $(d_2 + d_3 - 8 - a - c) + (d_2 + d_3 - 8 - b - d) - (d_2 - 5) = d_2 + 2d_3 - 11 - a - b - c - d = k + 2$, a contradiction. \hfill \Box

4. RESULTS: THE CASE $d_1 = 4$

We also studied the problem of decomposition into three factors with the smallest diameter 4. An early attempt to solve this problem is due to Řihová [6]. For $d_2$ equal to 4 and 5 she found the values of $F(4, d_2, d_3)$ and for $d_2 \geq 6$
she established the inequality
\[ F(4, d_2, d_3) \leq d_2 + d_3 - 4. \] (2)

We improve the upper bound (2) by constructing of decomposition of the graph \( K_{d_2 + d_3 - 5} \) into such three factors, proving thereby the following result.

**Theorem 2** Let \( d_2 \geq 6 \). Then \( F(4, d_2, d_3) \leq d_2 + d_3 - 5 \).

**Proof** It suffices to decompose the graph \( K_{d_2 + d_3 - 5} \) into three factors \( F_1, F_2, F_3 \) with the diameters \( d(F_1) = 4 \), \( d(F_2) = d_2 \) and \( d(F_3) = d_3 \). Let us denote the vertices of \( K_{d_2 + d_3 - 5} \) by \( u_1, u_2, ..., u_7, v_1, ..., v_{d_2-6}, w_1, ..., w_{d_3-6} \).

We first decompose \( K_7 \) with vertices \( u_1, u_2, ..., u_7 \) into the factors \( F'_1, F'_2, F'_3 \) with the diameters \( d(F'_1) = 4 \), \( d(F'_2) = 6 \) and \( d(F'_3) = 6 \). Define the sets of edges of \( F'_1, F'_2 \) and \( F'_3 \) as follows:

\[
E(F'_1) = \{u_1u_5, u_5u_7, u_7u_3, u_3u_2, u_6u_1, u_6u_5, u_6u_7, u_4u_3, u_4u_2\},
\]

\[
E(F'_2) = \{u_7u_4, u_4u_1, u_1u_2, u_2u_6, u_6u_3, u_3u_5\},
\]

\[
E(F'_3) = \{u_3u_1, u_1u_7, u_7u_2, u_2u_5, u_5u_4, u_4u_6\}.
\]

As the next step, we define the sets:

\[
X_2 = \emptyset \text{ if } d_3 = 6,
\]

\[
\{u_1w_i, u_2w_i, i = 1, ..., d_3 - 6\} \text{ if } d_3 \geq 7;
\]

\[
Y_2 = \emptyset \text{ if } d_2 = 6,
\]

\[
\{u_7v_1\}, \text{ if } d_2 = 7,
\]

\[
\{u_7v_1, v_1v_2, ..., v_{d_2-7}v_{d_2-6}\} \text{ if } d_2 \geq 8;
\]

\[
X_3 = \emptyset \text{ if } d_3 = 6,
\]

\[
\{u_3w_1\} \text{ if } d_3 = 7,
\]

\[
\{u_3w_1, w_1w_2, ..., w_{d_3-7}w_{d_3-6}\} \text{ if } d_3 \geq 8;
\]

\[
6
\]
\[ Y_3 = \emptyset \text{ if } d_2 = 6, \]
\[ \{ u_iv_i, u_2v_i, i = 1, 2, ..., d_2 - 6 \} \text{ if } d_2 \geq 7; \]
\[ X_1 = \{ u_iv_j, u_iw_k, v_jw_k, v_xv_s, w_yw_t, i = 3, 4, 5, 6, 7; j, s, x = 1, 2, ..., d_2 - 6, \]
\[ \text{ where } x \neq s; k, t, y = 1, 2, ..., d_3 - 6, \text{ where } y \neq t \} \cup (Y_2 \cup X_3). \]

Then the sets of edges of the factors \( F_1, F_2, F_3 \) of decomposition of \( K_{d_2+d_3-5} \) are:
\[ E(F_1) = E(F'_1) \cup X_1, \]
\[ E(F_2) = E(F'_2) \cup X_2 \cup Y_2, \]
\[ E(F_3) = E(F'_3) \cup X_3 \cup Y_3. \]

It is easy to check that \( d(F_2) = d_2 \) and \( d(F_3) = d_3 \). It remains to show that \( d(F_1) = 4 \). In \( F_1 \), the vertex \( u_1 \) is adjacent only to \( u_5 \) and \( u_6 \); and the vertex \( u_2 \) is adjacent only to \( u_3 \) and \( u_4 \). Note that no edge from the set \( \{ u_3u_5, u_3u_6, u_4u_5, u_4u_6 \} \) belongs to \( E(F_1) \), therefore \( d(F_1) \geq 4 \). Since every vertex \( v_j, j = 1, 2, ..., d_2 - 6 \), and every vertex \( w_k, k = 1, 2, ..., d_3 - 6 \), is adjacent to the vertices \( u_4, u_5, u_6 \), we have \( d(F_1) = 4 \). \( \square \)

Finally, we show that for \( d_2 \geq 6 \) the graph \( K_{d_2+d_3-6} \) cannot be decomposed into three factors.

**Theorem 3** Let \( d_2 \geq 6 \). Then \( F(4, d_2, d_3) = d_2 + d_3 - 5 \).

**Proof** By Theorem 2, we have \( F(4, d_2, d_3) \leq d_2 + d_3 - 5 \). If \( d_2 = 6 \), then \( F(4, 6, d_3) \leq d_3 + 1 \). It is evident that \( F(4, 6, d_3) > d_3 \), hence \( F(4, 6, d_3) = d_3 + 1 \).

Assume that \( d_2 \geq 7 \). We prove that the graph \( K_{d_2+d_3-6} \) cannot be decomposed into three factors with diameters 4, \( d_2 \) and \( d_3 \). Suppose the contrary and let \( K_{d_2+d_3-6} \) be decomposable into such factors. Let \( A_1 = u_1pqrw_2 \) be a fixed diameter path of \( F_1 \). According to Lemma A, the degree of each vertex in \( F_2 (F_3) \) is at most \( d_3 - 5 \) (\( d_2 - 5 \)), therefore the degree of each vertex in \( F_1 \) is at least \( (d_2 + d_3 - 7) - (d_2 - 5) - (d_3 - 5) = 3 \).

Consider the case \( d_2 = d_3 = 7 \) first. Since any vertex of \( F_1 \) is of degree at least 3, the vertex \( u_1 (u_2) \) is adjacent to at least two vertices not in \( A_1 \). We have vertices in \( F_1 \) not in \( A_1 \), hence there exists at least one vertex adjacent to both \( u_1 \) and \( u_2 \) which is absurd.

Suppose that \( d_2 \geq 7 \) and \( d_3 \geq 8 \). Denote the degree of \( u_i, i = 1, 2, \) in \( F_1 \) by \( \deg(u_1) = a \) and \( \deg(u_2) = b \), where \( a, b \geq 3 \). Since the number of vertices not in \( A_1 \) is \( d_2 + d_3 - 11 \), we have \( a + b \leq d_2 + d_3 - 9 \). There are \( d_2 + d_3 - 8 - (a + b) \) vertices adjacent neither to \( u_1 \) nor to \( u_2 \) in \( F_1 \).
We may assume without loss of generality that \( u_1 u_2 \in E(F_2) \). Let us denote the degree of \( u_1 \) and \( u_2 \) in \( F_2 \) by \( \text{deg}(u_1) = c \) and \( \text{deg}(u_2) = d \). Because every vertex of \( F_3 \) has degree at most \( d_2 = 5 \) and \( u_1 \) (\( u_2 \)) is of degree \( a \) (\( b \)) in \( F_1 \), obviously \( (d_2 + d_3 - 7) - (d_2 - 5) = a = d_3 - 2 - a \leq c = d_3 - 5 \) and consequently, \( d_2 + d_3 - 7 - (d_2 - 5) - b = d_3 - 2 - b \leq d \leq d_3 - 5 \) in \( F_2 \). It follows from part I of Lemma 1 that, if \( c + d > d_3 - 3 = (d_2 + d_3 - 6) - d_2 + 3 \) (\( c + d \leq d_3 - 3 \)), then there are at least \( c + d - (d_3 - 3) \) vertices (alternatively no vertex) adjacent to both \( u_1 \) and \( u_2 \) in \( F_2 \). It is evident that \( c + d - (d_3 - 3) \) has to be \( \leq d_2 + d_3 - 8 - (a + b) \). We distinguish two cases:

a) Let \( c + d - (d_3 - 3) < d_2 + d_3 - 8 - (a + b) \). Denote the number \( d_2 + d_3 - 8 - (a + b) - (c + d - d_3 + 3) = d_2 + 2d_3 - 11 - a - b - c - d \) by \( k \), where \( k > 0 \). In \( F_3 \) we have \( \text{deg}(u_1) = d_2 + d_3 - 7 - a - c \) and \( \text{deg}(u_2) = d_2 + d_3 - 7 - b - d \). If \( \text{deg}(u_1) + \text{deg}(u_2) \leq d_2 - 3 \), then \( 2(d_2 + d_3 - 7) - a - b - c - d = k \leq 0 \). On the other hand, if \( \text{deg}(u_1) + \text{deg}(u_2) > d_2 - 3 = (d_2 + d_3 - 6) - d_3 + 3 \), by part II of Lemma 1 it follows that the number of vertices adjacent to both \( u_1 \) and \( u_2 \) in \( F_3 \) is at least \( (d_2 + d_3 - 7 - a - c) + (d_2 + d_3 - 7 - b - d) - (d_3 - 4) = d_2 + 2d_3 - 10 - a - b - c - d = k + 1 \), which gives a contradiction.

b) Let \( c + d - (d_3 - 3) = d_2 + d_3 - 8 - (a + b) \). It is easy to see that the vertices adjacent to both \( u_1 \) and \( u_2 \) in \( F_2 \) are the same as the vertices adjacent neither to \( u_1 \) nor to \( u_2 \) in \( F_1 \). Denote by \( A_2 \) a fixed diameter path of \( F_2 \).

b1) Assume that the degrees of both end-vertices of \( A_2 \) are greater than 1. If both have distance \( \leq 2 \) from some vertex \( u_i \), \( i = 1, 2 \), then the diameter of \( F_2 \) can not be \( d_2 \), because \( u_1 u_2 \in E(F_2) \).

If at least one end-vertex of \( A_2 \) has distance greater than 2 from both \( u_1 \) and \( u_2 \), then there exists the vertex \( y \notin A_2 \) adjacent to some end-vertex of \( A_2 \), but adjacent neither to \( u_1 \) nor \( u_2 \) in \( F_2 \). Consequently, invoking part I of Lemma 1 it follows that the number of vertices adjacent to both \( u_1 \) and \( u_2 \) in \( F_2 \) is at least \( c + d - ((d_2 + d_3 - 6) - d_2 + 3) + 1 = c + d - (d_3 - 4) \). This is more than \( d_2 + d_3 - 8 - (a + b) \), a contradiction.

b2) Suppose that the degree of some end-vertex of \( A_2 \) is equal to 1. We denote such a vertex by \( x \). Now we prove that in spite of the fact that \( x \) is in \( F_2 \) adjacent to at most one vertex \( u_i \), \( i = 1 \) or 2, the vertex \( x \) is adjacent neither to \( u_1 \) nor \( u_2 \) in \( F_1 \), which will give a final contradiction. Let \( a = \text{deg}(u_1) \leq \text{deg}(u_2) = b \) in \( F_1 \). (The case \( \text{deg}(u_1) > \text{deg}(u_2) \) can be handled similarly.) We show that \( x \) can not be adjacent to \( u_2 \) in \( F_1 \).

A. If \( \text{deg}(x) < a \) in \( F_3 \), then \( \text{deg}(x) \geq d_2 + d_3 - 7 - a \) in \( F_1 \), which means
that in $F_1$, the vertex $x$ is not adjacent to at most $a$ vertices. But observe that any vertex adjacent to $u_2$ cannot be adjacent to $a+1$ vertices of distance $\leq 1$ from $u_1$ in $F_1$.

B. Let us assume that in $F_3$, $\deg(x) = a - 1 + z$, where $z \in N$. Then in $F_1$ we have $\deg(x) = (d_2 + d_3 - 7) - 1 - (a - 1 + z) = d_2 + d_3 - 7 - a - z$. By part II of Lemma 1, there exist at least $(a - 1 + z) + (d_2 + d_3 - 7 - a - c) - [(d_2 + d_3 - 6) - d_3 + 2] = d_3 - 4 - c + z$ vertices adjacent to both $u_1$ and $x$ in $F_3$. Since $c \leq d_3 - 5$, there are at least $z + 1$ such vertices. Suppose that $x$ is adjacent to $u_2$ in $F_1$. Then $x$ can not be adjacent to $a + 1$ vertices of distance $\leq 1$ from $u_1$ and to $z + 1$ vertices adjacent to both $u_1$ and $x$ in $F_3$. Because $\deg(x) = d_2 + d_3 - 7 - a - z$ in $F_1$, we get a contradiction.

One can show in a similar way that $x$ cannot be adjacent to $u_1$ in $F_1$, which completes the proof. \(\Box\)

The problem of decomposing a complete graph into three factors with the smallest diameter $d_1$ equal to 3 or 4 is therefore solved. In view of [4], for a complete determination of the values of $F(d_1, d_2, d_3)$ for $d_1 \leq d_2 \leq d_3 < \infty$ it remains to determine the exact values for $5 \leq d_1 \leq 65$.

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References


