DECOMPOSITIONS OF COMPLETE GRAPHS INTO THREE FACTORS

Tomáš Vetrík *

Decompositions of complete graphs into factors with given diameters are known to have the following hereditary property: if $K_n$ is decomposable into $m$ factors with diameters $d_1, d_2, \ldots, d_m$, then so is any $K_{n'}$ for $n' > n$. Let $F(d_1, d_2, \ldots, d_m)$ denote the smallest $n$ for which $K_n$ admits a decomposition into $m$ factors with diameters $d_1, d_2, \ldots, d_m$. We summarize results on decomposition of complete graphs into three factors and present new result regarding the value of $F(3, d_2, d_3)$.

Keywords: complete graphs, factors, decompositions of graphs into factors

2000 Mathematics Subject Classification: 05C10, 05C38

1 INTRODUCTION

All graphs considered in this paper are finite, undirected, without loops and multiple edges. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. By a factor of the graph $G$ we mean a subgraph of $G$ containing all vertices of $G$. A set $F_1, F_2, \ldots, F_m$ of factors of $G$ forms a factorization (a decomposition into factors) of $G$ if every edge of $G$ belongs to exactly one of the factors. A factorisation of a graph is isomorphic, if the factors are mutually isomorphic. The diameter $d(G)$ of a connected graph $G$ is the maximum distance between two vertices of $G$. For a disconnected graph $G$ we define $d(G) = \infty$.

The study of decompositions of complete graphs into factors with given diameters was initiated by Bosák, Rosa and Znám [1]. They proved the following fundamental result: if the complete graph $K_n$ is decomposable into $m$ factors $F_1, F_2, \ldots, F_m$ with prescribed diameters $d_1, d_2, \ldots, d_m$, then for every $n' > n$ the complete graph $K_{n'}$ is also decomposable into such factors.

We will briefly sum up the contents of this result by saying that decomposability of complete graphs into factors with given diameters is hereditary upwards. This property enables us to define the function $F(d_1, d_2, \ldots, d_m)$ to be the smallest positive integer $n$ such that the graph $K_n$ can be decomposed into $m$ factors with prescribed diameters $d_1, d_2, \ldots, d_m$. If such an integer does not exist we set $F(d_1, d_2, \ldots, d_m) = \infty$. If all the factors have the same diameter $d$, we write $F(d_1, d_2, \ldots, d_m) = F_m(d)$.

2 DECOMPOSITION INTO 3 FACTORS

We begin with presenting known facts on decomposition of complete graphs into 3 factors and then we provide some new results. Let us assume that the diameters $d_1, d_2, d_3$ satisfy $d_1 \leq d_2 \leq d_3 < \infty$.

Bosák, Rosa and Znám [1] proved the following theorem.

Theorem A. Let $d_1 \geq 5$.
Then $F(d_1, d_2, d_3) \leq d_1 + d_2 + d_3 - 8$.

Later, Hrnčiar [2] proved that the equality in Theorem A holds for any $d_1$, sufficiently large.

Theorem B. Let $d_1 > 65$.
Then $F(d_1, d_2, d_3) = d_1 + d_2 + d_3 - 8$.

It makes sense to study small values of $d_1$ and this has been done by a number of authors. Bosák, Rosa and Znám in [1] gave a complete solution to the problem of decomposing a complete graph into three factors with diameters $2, d_2, d_3$ except for the exact value of $F(2, 2, 2)$, where they ended up with the inequality $12 \leq F(2, 2, 2) \leq 13$.

The exact value was found by Stacho and Urland with the help of a computer search. In [4] they proved that $K_{12}$ is not decomposable into three factors with diameters equal to 2 and hence $F(2, 2, 2) = 13$.

Palumbíný [3] studied decomposition into 3 factors with the smallest diameter $d_1 = 3$. For small $d_2$ and arbitrary $d_3$ he found the values of the function $F(3, d_2, d_3)$ and for $d_2 \geq 9$ he proved that $F(3, d_2, d_3) \leq d_2 + d_3 - 6$.

We have made a contribution to the case $d_1 = 3$. Our result is that the complete graph of order $d_2 + d_3 - 7$ is not decomposable into 3 factors with diameters $3, d_2$ and $d_3$ where $d_2 \geq 9$ and therefore $F(3, d_2, d_3) = d_2 + d_3 - 6$.

This is the contents of Theorem 1; we precede its formulation by an auxiliary result that is interesting on its own and is used in the proof of Theorem 1.

* Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia, E-mail: vetrik@math.sk

ISSN 1335-3632 © 2005 FEI STU
Theorem 1. Let $u, v$ be two distinct vertices of a graph $G$ with $n$ vertices ($n \geq 5$) and a finite diameter $d$ ($d \geq 2$). Let $\deg(u) = a$, $\deg(v) = b$ and $a + b > n - d + 3$.

I. If the edge $uv \in E(G)$ then there exists at least $(a+b)-(n-d+3)$ vertices adjacent to both vertices $u$ and $v$ in $G$.

II. If the edge $uv \notin E(G)$ then there exists at least $(a+b)-(n-d+2)$ vertices adjacent to both vertices $u$ and $v$ in $G$.

Proof. Let $a + b = n - d + 4 + k$ where $k \in N$, $k \geq 0$. The set of all vertices adjacent to vertex $u$ ($v$) will be denoted by $A$ ($B$).

I. Let $uv \in E(G)$. We first prove that $|A \cup B| \leq n - d + 3$. Let $A_1$ be the fixed diameter path in $G$. We consider 3 cases:

a) Let $u, v \in A_1$. Both $u$ and $v$ can be adjacent to at most two vertices from $A_1$ and $n-d-1$ vertices not in $A_1$. Thus $|A \cup B| \leq (n-d-1) + 2 + 2 = n - d + 3$.

b) Let $u \in A_1$, $v \notin A_1$. Besides $u$, the vertex $v$ is adjacent to at most two vertices $v_1, v_2 \in A_1$ (possibly $v_1 = v_2$). The vertex $u$ is adjacent to at most two vertices $u_1, u_2 \in A_1$. If $v_1 \neq v_2$ (since $uv \in E(G)$), at least one of them must coincide with some vertex $u_i$, $i \in \{1, 2\}$. Therefore, there exist at most 4 vertices in $A_1$ and $n-d-1$ vertices not in $A_1$, contained in $A \cup B$. It follows that $|A \cup B| \leq n - d + 3$.

c) Let $u \notin A_1$, $v \notin A_1$. The vertex $u$ ($v$) is adjacent to at most three vertices from $A_1$, we denote them by $u_i, 1 \leq i \leq 3$, ($v_j, 1 \leq j \leq 3$). Again, we prove that there exist at most four distinct vertices in $A_1$ contained in $A \cup B$. If $v_i \neq v_j$ (since $uv \in E(G)$), at least one of them must coincide with some vertex $u_i$, $i \in \{1, 2, 3\}$, then there exist vertices $u_i, v_j$, where $i, j \in \{1, 2, 3\}$ such that in $A_1$, $\rho(u_i, v_j) = 4$. But there exists a path $u_iuv_j$ of length 3 in $G$ and it is a contradiction with equality $\rho(u_i, v_j) = 4$.

From the facts that $a + b = |A| + |B| = n - d + 4 + k$ and $|A \cup B| \leq n - d + 3$ it follows that $|A \cap B| = |A| + |B| - |A \cup B| \geq (n-d+4+k) - (n-d+3) = (a+b)-(n-d+3)$. This finishes the first part of the proof.

II. Let $uv \notin E(G)$. Similarly, we can distinguish three cases and prove that $|A \cap B| \geq (a+b)-(n-d+2)$.

Theorem 1. Let $d_2 \geq 9$. Then $F(3, d_2, d_3) = d_2 + d_3 - 6$.

Proof. To prove the assertion we have to show that the graph $K_{d_2+d_3-7}$ is not decomposable into three factors with diameters 3, $d_2$ and $d_3$. We prove this by contradiction. Suppose that such a decomposition exists. Let $F_1 = u_1xu_2$ be the fixed diameter path of the factor $F_1$. Since the factor $F_2$ ($F_3$) has diameter $d_2$ ($d_3$), the degree of every vertex in $F_2$ is at most $d_2 + d_3 - 7 - d_2 + 1 = d_3 - 6$ (in $F_3$ at most $d_2 + d_3 - 7 - d_3 + 1 = d_2 - 6$). Hence every vertex of $F_1$ has the degree $\geq 4 = d_2 + d_3 - 8 - (d_2 - 6) = (d_3 - 6)$. Let us denote the degree of the vertex $u_i, i = 1, 2$; in $F_1$ by $\deg(u_1) = a$ and $\deg(u_2) = b$. Here $a \geq 4$, $b \geq 4$ and $a + b \leq d_2 + d_3 - 9$, whereas the number of vertices not in $A_1$ is $d_2 + d_3 - 11$. There are $d_2 + d_3 - 9 - (a+b)$ vertices adjacent neither to $u_1$ nor to $u_2$ in $F_1$. Without lost of generality we may suppose that $u_1u_2 \in F_2$. We denote the degree of $u_i, i = 1, 2$; in $F_2$ by $\deg(u_1) = c$ and $\deg(u_2) = d$. In $F_2$, the degree of every vertex is at most $d_2$ and in $F_1$, the degree of vertex $u_3$ ($u_2$) is $a$ ($b$). Hence in $F_2$, $d_2 + d_3 - 8 - (d_2 - 6) = d_3 - 2 - a \leq c \leq d_3 - 6$ ($d_2 + d_3 - 8 - (d_2 - 6) = d_3 - 2 - b \leq d \leq d_3 - 6$).

It follows from part I. of Lemma 1 that if $c + d > d_2 + d_3 - 7 - d_2 + 3 = d_3 - 4$ then there exist at least $c + d - (d_3 - 4)$ vertices (alternatively no vertex at all) adjacent to both vertices $u_1$ and $u_2$ in $F_2$. If $c + d - (d_3 - 4) > d_2 + d_3 - 9 - (a+b)$, we have a contradiction.

Hence we suppose that $c + d - (d_3 - 4) \leq d_2 + d_3 - 9 - (a+b)$. Denote the quantity $d_2 + d_3 - 9 - (a+b) - (c + d - (d_3 - 4)) = d_2 + 2d_3 - 13 - a - b - c - d$ by $k$, clearly $k \geq 0$.

With the help of part II. of Lemma 1 it can be proved that the number of vertices adjacent to both vertices $u_1$ and $u_2$ in $F_2$ exceeds $k$, which gives a contradiction.

The problem of decomposition of complete graph into 3 factors with the smallest diameter $d_1 = 3$ is solved completely.

Acknowledgment

The author acknowledges many valuable suggestions of Professor Jozef Sirán during preparation of the paper.

REFERENCES


Received 16 June 2005

Tomáš Vetřík (RNDr) is a PhD student at the Faculty of Civil Engineering of the Slovak University of Technology. His supervisor (in applied mathematics) is Professor Jozef Sirán.