Computing the metric dimension of the categorial product of some graphs

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A set of vertices $W$ is a resolving set of a graph $G$ if every two vertices of $G$ have distinct representations of distances with respect to the set $W$. The number of vertices in a smallest resolving set is called the metric dimension. This invariant has extensive applications in robotics, since the metric dimension can represent the minimum number of landmarks, which uniquely determine the position of a robot moving in a graph space. Finding the metric dimension of a graph is an NP-hard problem. We present exact values of the metric dimension of several networks, which can be obtained as categorial products of graphs.

Keywords: metric dimension; resolving set; categorial product; robotics

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1. Introduction

Consider the following problem, which was studied in [6]. A robot moves in a space, which is modelled by a graph. The robot moves from a node to a node, and it can locate itself by the presence of distinctively labelled landmark nodes. The position of robot is represented by its distances to a set of landmarks. The problem is to find the minimum number of landmarks required, and to find out where they should be placed, such that the robot can always determine its location. In graph theory language, a minimum set of landmarks which uniquely determine the position of robot is called a metric basis, and the minimum number of landmarks is called the metric dimension.

The concept of metric dimension was introduced by Slater [10] and studied independently by Harary and Melter [3]. Slater referred to a metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of minimum number of loran/sonar detecting devices in a network so that the position of every vertex in the network can be uniquely represented in terms of its distances to the devices in the set. Applications of the study of metric dimension to the problem of pattern recognition and image processing are given in [7].

Let $G$ be a connected graph with vertex set $V(G)$. The distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path between them. A vertex $w$ resolves a pair of vertices $u, v$ if $d(u, w) \neq d(v, w)$. For an ordered set of vertices $W = \{w_1, w_2, \ldots, w_z\}$, the representation of distances of a vertex $v$ with respect to $W$ is

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the ordered \( z \)-tuple
\[
r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_z)).
\]

A set of vertices \( W \subset V(G) \) is a resolving set of \( G \) if every two vertices of \( G \) have distinct representations (if every pair of vertices of \( G \) is resolved by some vertex of \( W \)). The cardinality of a smallest resolving set is called the metric dimension and it is denoted by \( \text{dim}(G) \). Note that the \( i \)-th coordinate in \( r(v|W) \) is 0 if and only if \( v = w_i \). This means that in order to show that \( W \) is a resolving set of \( G \), it suffices to verify that \( r(u|W) \neq r(v|W) \) for every pair of distinct vertices \( u, v \in V(G) \setminus W \).

The metric dimension of various classes of graphs has been investigated for four decades. From [2] it follows that the question whether the metric dimension of a graph is less than a given value, is an NP-complete problem. In [5] and [9] the authors considered the metric dimension of the lexicographic product of graphs and results on the metric dimension of the corona product of graphs were presented by Iswadi, Baskoro and Simanjuntak [4]. Cáceres et. al. [1] studied the Cartesian product \( G \square H \) of two graphs \( G \) and \( H \). They proved that for all \( m, n \geq 3 \),

\[
\text{dim}(C_m \square C_n) = \begin{cases} 3, & \text{if } m \text{ or } n \text{ is odd} \\ 4, & \text{otherwise,} \end{cases}
\]

and for all \( m \geq 2 \) and \( n \geq 3 \), we have

\[
\text{dim}(P_m \square C_n) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 3, & \text{if } n \text{ is even}, \end{cases}
\]

where \( C_n \) is the cycle of order \( n \) and \( P_m \) is the path of order \( m \). Naeem and Imran [8] used a different technique to prove this result and they also studied the metric dimension of the Cartesian product of the path and the square of the cycle. They showed that for \( n \geq 5 \),

\[
\text{dim}(P_m \square C_n^2) = \begin{cases} 3, & \text{if } n \equiv 0, 2, 3 \pmod{4} \\ \leq 4, & \text{otherwise.} \end{cases}
\]

We study the connectivity and the metric dimension of interesting networks, which can be obtained as categorial products of graphs. We show that results on the metric dimension of categorial products of \( C_m \times C_n \) and \( P_m \times C_n \) for small values of \( m \) differ considerably from the metric dimension of Cartesian products of these graphs.

2. Connectivity of the categorial product of cycles and paths

The categorial product \( G \times H \) of two graphs \( G \) and \( H \) has the vertex set \( V(G) \times V(H) \), where two vertices \((u, u')\) and \((v, v')\) of \( G \times H \) are adjacent if and only if \( u, v \) are adjacent in \( G \) and \( u', v' \) are adjacent in \( H \).

Let \( C_n \) be the cycle, such that the vertices of \( C_n \) are the elements of the cyclic group \( \mathbb{Z}_n \) and \( E(C_n) = \{i, i+1 \mid i = 0, 1, \ldots, n-1\} \). Then the vertices of \( C_m \times C_n \) are the elements of the direct product of the groups \( \mathbb{Z}_m \) and \( \mathbb{Z}_n \), and \( E(C_m \times C_n) = \{(i, j)(i+1, j+1) \mid i, j = 0, 1, \ldots, m-1, n-1\} \).
1), (i, j)(i + 1, j − 1) \mid i = 0, 1, \ldots, m − 1; j = 0, 1, \ldots, n − 1}. The graph \( C_m \times C_n \) has order \( mn \) and size \( 2mn \). We study the connectivity of \( C_m \times C_n \).

**Lemma 2.1** Let \( m, n \geq 2 \). The graph \( C_m \times C_n \) is connected if and only if \( m \) or \( n \) is odd.

**Proof.** Let \( m \) and \( n \) be even. Let \((i, j)\) be any vertex of \( C_m \times C_n \), \( 0 \leq i \leq m − 1, 0 \leq j \leq n − 1 \). Since \((i, j)\) is adjacent only to the vertices \((i + 1, j + 1), (i + 1, j − 1), (i − 1, j + 1), (i − 1, j − 1)\), the vertex \((i, j)\) is joined by a path (or a walk) only to the vertices \((i + k, j + l)\), where the integers \( k, l \) are either both even or both odd. If \( i + k \) is even (odd), then \((i + k) (mod m)\) is even (odd), and similarly if \( j + l \) is even (odd), then \((j + l) (mod n)\) is even (odd). This implies that if exactly one of \( k', l' \) is even, then the vertices \((i, j)\) and \((i + k', j + l')\) are in different components.

Let \( n \) be odd. Since for any \( i = 0, 1, \ldots, m − 1 \), there is a cycle \((i, 0), (i + 1, 1), (i + 1, 3), \ldots, (i, n − 3), (i + 1, n − 2), (i, n − 1), (i + 1, 0), (i, 1), \ldots, (i, n − 3), (i, n − 2), (i + 1, n − 1)\) of length \( 2n \) in \( C_m \times C_n \), all vertices \((i, j), (i + 1, j)\) where \( j = 0, 1, \ldots, n − 1 \), are in one component. Hence \( C_m \times C_n \) is connected.

Let \( P_m \) be the path having the vertex set \( V(P_m) = \{0, 1, \ldots, m − 1\} \) and the edge set \( E(P_m) = \{i, i + 1 \mid i = 0, 1, \ldots, m − 2\} \). Then \( V(P_m \times C_n) = \{(i, j) \mid i = 0, 1, \ldots, m − 1; j = 0, 1, \ldots, n − 1\} \) and \( E(P_m \times C_n) = \{(i, j)(i + 1, j + 1), (i, j)(i + 1, j − 1) \mid i = 0, 1, \ldots, m − 2; j = 0, 1, \ldots, n − 1\} \). Note that the second index of any vertex of \( P_m \times C_n \) is taken modulo \( n \). The graph \( P_m \times C_n \) has order \( mn \) and size \( 2(m − 1)n \).

**Lemma 2.2** Let \( m \geq 2, n \geq 3 \). The graph \( P_m \times C_n \) is connected if and only if \( n \) is odd.

**Proof.** Let \( n \) be even. Since for \( 0 \leq j \leq n − 1 \) the vertex \((0, j)\) is adjacent only to the vertices \((1, j + 1), (1, j − 1)\), the vertex \((m − 1, j)\) is adjacent only to \((m − 2, j + 1), (m − 2, j − 1)\), and the vertex \((i, j)\) where \( 1 \leq i \leq m − 2 \), is adjacent to \((i + 1, j + 1), (i + 1, j − 1), (i − 1, j + 1), (i − 1, j − 1)\), there is a path between the vertices \((0, 0)\) and \((i, j)\) only if both \( i, j \) are even, or both \( i, j \) are odd, \( i = 0, 1, \ldots, m − 1, j = 0, 1, \ldots, n − 1 \). If \( j \) is even (odd), then \((j) (mod n)\) is even (odd), therefore \((0, 0)\) cannot be joined by a path to a vertex \((i', j')\), where exactly one of \( i', j' \) is even.

Let \( n \) be odd. For any \( i = 0, 1, \ldots, m − 2 \), there is a cycle of length \( 2n \) containing all vertices \((i, j), (i + 1, j)\) where \( j = 0, 1, \ldots, n − 1 \), which implies that \( P_m \times C_n \) is connected.

Now we consider the connectivity of the categorial product of two paths. Let \( V(P_m \times P_n) = \{(i, j) \mid i = 0, 1, \ldots, m − 1; j = 0, 1, \ldots, n − 1\} \). Then \( E(P_m \times P_n) = \{(i, j)(i + 1, j + 1), (i, j)(i + 1, j − 1) \mid i = 0, 1, \ldots, m − 2; j = 0, 1, \ldots, n − 2\} \) \( \cup \{(i, j)(i + 1, j − 1) \mid i = 0, 1, \ldots, m − 2; j = 1, 2, \ldots, n − 1\} \). Let us note that the size of \( P_m \times P_n \) is \( 2(m − 1)(n − 1) \).

**Lemma 2.3** Let \( m, n \geq 2 \). The graph \( P_m \times P_n \) consists of two components.

**Proof.** From the definition of the graph \( P_m \times P_n \) it follows that the vertex \((0, 0)\) is in the same component only with the vertices \((i, j)\), where either both \( i, j \) are even, or both \( i, j \) are odd, \( 0 \leq i \leq m − 1, 0 \leq j \leq n − 1 \). Similarly the vertices \((i, j)\), such that exactly one of \( i, j \) is even, form the other component of \( P_m \times P_n \).

Since the graph \( P_m \times P_n \) is disconnected, \( \text{dim}(P_m \times P_n) = \infty \) for any \( m, n \geq 2 \). In the next section we consider the metric dimension of the graphs \( C_m \times C_n \) and \( C_m \times P_n \).
3. Main results

From Lemma 2.1 it follows that \( \text{dim}(C_m \times C_n) = \infty \) if both \( m \) and \( n \) are even. We study the metric dimension of the graphs \( C_m \times C_n \) for \( m = 3 \). Let us denote the vertices of \( C_3 \times C_n \) by \( v_{i,j} \) (instead of \( (i,j) \) which is used in the previous section), \( i = 0, 1, 2; \ j = 0, 1, \ldots, n - 1 \). The first index of any vertex \( v_{i,j} \) is taken modulo 3 and the second index is taken modulo \( n \).

**Theorem 3.1** Let \( n \geq 9 \). Then \( \text{dim}(C_3 \times C_n) = \lfloor \frac{2n}{3} \rfloor \).

**Proof.** Let us present the distances between all pairs of vertices of \( C_3 \times C_n \). For any vertex \( v_{i,j} \in V(C_3 \times C_n) \), \( i = 0, 1, 2; \ j = 0, 1, \ldots, n - 1 \), we have

\[
\begin{align*}
d(v_{i,j}, v_{i+1,j}) &= d(v_{i,j}, v_{i+2,j}) = 2, \\
d(v_{i,j}, v_{i,j+1}) &= 3, \\
d(v_{i,j}, v_{i+1,j+1}) &= d(v_{i,j}, v_{i+2,j+1}) = 1, \\
d(v_{i,j}, v_{i,j+1+\epsilon}) &= d(v_{i,j}, v_{i+1,j+1+\epsilon}) = d(v_{i,j}, v_{i+2,j+1+\epsilon}) = p,
\end{align*}
\]

where \( \epsilon = \pm 1 \) and \( p = 2, 3, \ldots, \lfloor n/2 \rfloor \).

Let \( V_j = \{v_{0,j}, v_{1,j}, v_{2,j}\} \), \( j = 0, 1, \ldots, n - 1 \) and \( U_j = V_{j-1} \cup V_j \cup V_{j+1} \), where the indices are taken modulo \( n \).

First we prove that \( \text{dim}(C_3 \times C_n) \geq \lfloor \frac{2n}{3} \rfloor \). Let \( W \) be any resolving set of the graph \( C_3 \times C_n \). We prove that \( |W| \geq \frac{2n}{3} \) by showing that \( |U_j \cap W| \geq 2 \) for any \( j = 0, 1, \ldots, n - 1 \). Note that for any vertex \( u' \notin U_j \), we have

\[
d(u', v_{0,j}) = d(u', v_{1,j}) = d(u', v_{2,j}),
\]

which means that \( u' \) cannot resolve any vertices of \( V_j \). For any vertex \( u \in U_j \) there are two vertices in \( V_j \), which are not resolved by \( u \). Therefore in order to resolve all 3 vertices of \( V_j \), the set \( U_j \) contains at least 2 vertices of \( W \). Thus \( |W| \geq 2n/3 \). Since \( |W| \) is an integer, we get \( |W| \geq \lfloor \frac{2n}{3} \rfloor \).

Now we show that there exists a resolving set of \( C_3 \times C_n \) having \( \lfloor \frac{2n}{3} \rfloor \) vertices. We distinguish two cases.

**Case 1:** \( n \equiv 0 \) or \( 2 \pmod{3} \).

Let \( n = 3k \) if \( n \equiv 0 \pmod{3} \) and \( n = 3k + 2 \) if \( n \equiv 2 \pmod{3} \), \( k \geq 3 \). Let \( W = \{v_{0,j}, v_{1,j} \mid j = 1, 4, 7, \ldots, 3k + c\} \), where \( c = -2 \) if \( n = 3k \), and \( c = 1 \) if \( n = 3k + 2 \). We prove that \( W \) is a resolving set of \( C_3 \times C_n \). Let us present the representations of distances of \( V(C_3 \times C_n) \setminus W_j \) with respect to \( W_j = \{v_{0,j}, v_{1,j}\} \subset W \), where \( j = 1, 4, 7, \ldots, 3k + c \).

We have

\[
\begin{align*}
r(v_{2,j}|W_j) &= (2, 2), \\
r(v_{0,j+\epsilon}|W_j) &= (3, 1), \quad r(v_{1,j+\epsilon}|W_j) = (1, 3), \quad r(v_{2,j+\epsilon}|W_j) = (1, 1), \\
r(v_{0,j+\epsilon p}|W_j) &= r(v_{1,j+\epsilon p}|W_j) = r(v_{2,j+\epsilon p}|W_j) = (p, p),
\end{align*}
\]

where \( \epsilon = \pm 1 \) and \( p = 2, 3, \ldots, \lfloor n/2 \rfloor \).

We show that for any \( j = 1, 4, 7, \ldots, 3k - 2 \), all vertices of \( U_j \) are resolved by \( W_j \cup W_{j+3} \). The following pairs of vertices of \( U_j \) are not resolved by \( W_j \): \( v_{0,j-1}, v_{0,j+1}; v_{1,j-1}, v_{1,j+1}; v_{2,j-1}, v_{2,j+1}; \) and the vertex \( v_{2,j} \) has the same representation as the vertices of \( V_{j-2} \) and
The vertex $v_{2j}$ is also resolved, since
\[
    d(v_{2j}, v_{0,j+3}) = 3,
    d(v_{0,j+2}, v_{1,j+3}) = d(v_{2j+2}, v_{0,j+3}) = 1,
\]
and the distance between $v_{0,j+3}$ and any vertex in $V_{j-2}$ is greater than 3.

Equivalently it can be shown that if $n = 3k + 2$, then the vertices of $U_{3k+1}$ are resolved by $W_{3k+1} \cup W_{3k-2}$.

Since $U_1 \cup U_2 \cup \cdots \cup U_{3k+c} = V(C_3 \times C_n)$, all vertices of $C_3 \times C_n$ are resolved by $W$.

**Case 2:** \( n \equiv 1 \pmod{3} \).

Let us show that $W = \{v_{0,j-1}, v_{1j} \mid j = 1, 4, 7, \ldots, n - 3\} \cup \{v_{2,n-1}\}$ is a resolving set of $C_3 \times C_n$. Note that $|W| = (n-1)/3 + 1 = [3n/2]$. We give the representations of distances of $V(C_3 \times C_n) \setminus W_j$ with respect to $W_j = \{v_{0,j-1}, v_{1j}\} \subset W$, where $j = 1, 4, 7, \ldots, n - 3$:
\[
    r(v_{0,j-2}|W_j) = r(v_{0,j}|W_j) = (3, 2),
    r(v_{1,j-1}|W_j) = r(v_{1,j+1}|W_j) = (2, 3),
    r(v_{1,j-2}|W_j) = r(v_{2,j-2}|W_j) = r(v_{2j}|W_j) = (1, 2),
    r(v_{0,j+1}|W_j) = r(v_{2j+1}|W_j) = r(v_{2j-1}|W_j) = (2, 1),
    r(v_{0,j+p}|W_j) = r(v_{1,j+p}|W_j) = r(v_{2,j+p}|W_j) = (p+1, p),
    r(v_{0,j-p-1}|W_j) = r(v_{1,j-p-1}|W_j) = r(v_{2j-p-1}|W_j) = (p, p+1),
\]
where $p = 2, 3, \ldots, [n/2] - 1$. If $n$ is odd, then
\[
    r(v_{0,j+(n-1)/2}|W_j) = r(v_{1,j+(n-1)/2}|W_j) = r(v_{2j+(n-1)/2}|W_j) = ((n-1)/2, (n-1)/2).
\]

We show that for any $j = 1, 4, 7, \ldots, n - 6$, all vertices of $U_j$ are resolved by $W$. We present the vertices of $U_j$, which have the same representations (with respect to $W_j$) with any vertices of $C_3 \times C_n$: \((v_{0,j}, v_{0,j-2}, v_{2j-2}), (v_{1,j-1}, v_{1j+1}, v_{x,j}), (v_{0,j+1}, v_{2j-1}, v_{2j-2}), (v_{0,j+1}, v_{2j+1}, v_{2j+1})\) where $x = 0, 1, 2$. The vertex $v_{0,j+2}$ resolves most of the vertices of $U_j$, which are not resolved by $W_j$, since
\[
    d(v_{0,j}, v_{0,j+2}) = d(v_{1,j+2}, v_{0,j+2}) = d(v_{2j+2}, v_{0,j+2}) = 2, 
    d(v_{0,j-2}, v_{0,j+2}) = 4,
    d(v_{1,j-1}, v_{0,j+2}) = 3, 
    d(v_{1,j+1}, v_{0,j+2}) = 1, 
    d(v_{2j-3}, v_{0,j+2}) = 5,
    d(v_{2j}, v_{0,j+2}) = 2, 
    d(v_{1,j-2}, v_{0,j+2}) = d(v_{2j-2}, v_{0,j+2}) = 4,
    d(v_{0,j+1}, v_{0,j+2}) = d(v_{2j-1}, v_{0,j+2}) = 3, 
    d(v_{2j+1}, v_{0,j+2}) = 1.
\]

It remains to resolve the pairs $(v_{0,j}, v_{1j+2})$, $(v_{0,j}, v_{2j+2})$ and $(v_{0,j+1}, v_{2j+1})$. However these pairs are resolved by any vertex of $W_i$, where $i \in \{1, 4, 7, \ldots, n - 3\}$, $i \neq j$, $i \neq j + 3$. It follows that no vertex in $V_0 \cup V_1 \cup \cdots \cup V_{n-5}$ has the same representation of distances (with respect to $W$) with any other vertex of $C_3 \times C_n$.

It order to complete the proof we must show that no two vertices of $V_{n-4} \cup V_{n-3}$
$V_{n-2} \cup V_{n-1} \setminus W$ have the same representations of distances with respect to $W$. We have

\[
\begin{align*}
    r(v_{0,n-3}|W_{n-3}) &= r(v_{0,n-1}|W_{n-3}) = r(v_{1,n-1}|W_{n-3}) = (3, 2), \\
    r(v_{1,n-4}|W_{n-3}) &= r(v_{1,n-2}|W_{n-3}) = (2, 3), \\
    r(v_{2,n-3}|W_{n-3}) &= (1, 2), \\
    r(v_{0,n-2}|W_{n-3}) &= r(v_{2,n-2}|W_{n-3}) = r(v_{2,n-4}|W_{n-3}) = (2, 1).
\end{align*}
\]

The vertices which are not resolved by $W_{n-3}$, can be resolved by other vertices of $W$:

\[
\begin{align*}
    d(v_{0,n-3}, v_{0,0}) &= d(v_{0,n-1}, v_{0,0}) = 3, \quad d(v_{1,n-1}, v_{0,0}) = 1, \\
    d(v_{0,n-3}, v_{1,1}) &= 4, \quad d(v_{0,n-1}, v_{1,1}) = 2, \\
    d(v_{1,n-4}, v_{2,n-1}) &= 3, \quad d(v_{1,n-2}, v_{2,n-1}) = 1, \\
    d(v_{2,n-2}, v_{2,n-1}) &= d(v_{2,n-4}, v_{2,n-1}) = 3, \quad d(v_{0,n-2}, v_{2,n-1}) = 1, \\
    d(v_{2,n-2}, v_{0,0}) &= 2, \quad d(v_{2,n-4}, v_{0,0}) = 4.
\end{align*}
\]

Hence all vertices of $C_3 \times C_n$ have different representations of distances with respect to $W$. The proof is complete.

Resolving sets of $C_3 \times C_n$ for $3 \leq n \leq 8$ are given in Table 1.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Resolving set</th>
<th>Cardinality of resolving set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_3 \times C_3$</td>
<td>${v_{0,0}, v_{1,0}, v_{0,1}}$</td>
<td>3</td>
</tr>
<tr>
<td>$C_3 \times C_4$</td>
<td>${v_{0,0}, v_{1,0}, v_{2,0}, v_{0,1}, v_{1,1}, v_{2,1}}$</td>
<td>6</td>
</tr>
<tr>
<td>$C_3 \times C_5$</td>
<td>${v_{0,0}, v_{1,0}, v_{0,1}, v_{1,2}}$</td>
<td>4</td>
</tr>
<tr>
<td>$C_3 \times C_6$</td>
<td>${v_{0,0}, v_{1,0}, v_{2,1}, v_{0,3}, v_{1,3}, v_{2,4}}$</td>
<td>6</td>
</tr>
<tr>
<td>$C_3 \times C_7$</td>
<td>${v_{0,0}, v_{1,1}, v_{0,3}, v_{1,4}, v_{2,5}}$</td>
<td>5</td>
</tr>
<tr>
<td>$C_3 \times C_8$</td>
<td>${v_{0,0}, v_{1,1}, v_{0,3}, v_{1,4}, v_{0,6}, v_{1,7}}$</td>
<td>6</td>
</tr>
</tbody>
</table>

Now we consider the metric dimension of $P_m \times C_n$ for small values of $m$. Since $P_2 \times C_n$ $(n \geq 3)$ consists of two cycles having $n$ vertices if $n$ is even, and $P_2 \times C_n$ is the cycle with $2n$ vertices if $n$ is odd, we have $\dim(P_2 \times C_n) = \infty$ if $n$ is even, and $\dim(P_2 \times C_n) = 2$ if $n$ is odd.

Let us present exact values of the metric dimension of the graphs $P_m \times C_n$ for $m = 3$ and 4. We find useful to denote the vertices of $P_3 \times C_n$ and $P_4 \times C_n$ as follows: $V(P_3 \times C_n) = \{a_i, b_i, c_i \mid i = 0, 1, 2, \ldots, n - 1\}$ and $V(P_4 \times C_n) = \{a_i, b_i, c_i, e_i \mid i = 0, 1, 2, \ldots, n - 1\}$, where the indices $i$ of the vertices $a_i, b_i, c_i, e_i$ are elements of the cyclic group of order $n$, therefore they are taken modulo $n$. For the edge sets of $P_3 \times C_n$ and $P_4 \times C_n$ we have

$E(P_3 \times C_n) = \{a_ib_{i+1}, a_ib_{i-1}, b_ic_{i+1}, b_ic_{i-1} \mid i = 0, 1, 2, \ldots, n - 1\}$ and $E(P_4 \times C_n) = \{a_ib_{i+1}, a_ib_{i-1}, b_ic_{i+1}, b_ic_{i-1}, c_ie_{i+1}, c_ie_{i-1} \mid i = 0, 1, 2, \ldots, n - 1\}$, respectively.

**Theorem 3.2** Let $n \geq 5$ be odd. Then $\dim(P_3 \times C_n) = n$.

**Proof.** Let $x = a_i$ or $c_i$. It can be checked that

\[
\begin{align*}
    d(x, a_{i+p}) &= d(x, c_{i+p}) = p, \quad d(x, a_{i+s}) = d(x, c_{i+s}) = n - s, \quad d(a_i, c_i) = 2, \\
    d(x, b_{i+s}) &= s, \quad d(x, b_{i+p}) = n - p, \quad d(x, b_i) = n, \\
    d(b_i, b_{i+p}) &= p, \quad d(b_i, b_{i+s}) = n - s,
\end{align*}
\]

where \( p = \pm 2, \pm 4, \ldots, \pm t_1; \ s = \pm 1, \pm 3, \ldots \pm t_2; \ t_1, t_2 \in \{(n - 3)/2, (n - 1)/2\} \) (\( t_1 \) is even, \( t_2 \) is odd).

We show that we need at least \( n \) vertices to resolve all pairs \( a_j, c_j \) for \( j = 0, 1, 2, \ldots, n - 1 \). No vertex \( b_i \), \( i = 0, 1, 2, \ldots, n - 1 \), can resolve the vertices \( a_j, c_j \), since \( d(b_i, a_j) = d(b_i, c_j) \). The vertex \( a_0 \) resolves only the pair \( a_0, c_1 \), and it does not resolve the pair of vertices \( a_j, c_j \) for \( j = 0, 1, 2, \ldots, n - 1, j \neq i \). Therefore any resolving set of \( P_3 \times C_n \) contains at least \( n \) vertices and we get \( \text{dim}(P_3 \times C_n) \geq n \).

Now we prove that the set of \( n \) vertices \( W = \{a_0, a_1, \ldots, a_{n-1}\} \) resolves all the other vertices of \( P_3 \times C_n \) (in fact even much smaller set resolves the vertices in \( V(P_3 \times C_n) \setminus W \)). We give the representation of \( d(x, e) \) for \( x \in \{a_0, a_1, \ldots, a_{n-1}\}, e \in \{b_1, b_2, \ldots, n\} \). We have

\[
\begin{align*}
r(b_i | W) &= (i, i - 2), \quad i = 3, 5, \ldots, n, \\
r(b_{i-1} | W) &= (i, i + 2), \quad i = 1, 3, \ldots, n - 2, \\
r(c_i | W) &= (i, i - 2), \quad i = 2, 4, \ldots, n - 4, \\
r(b_{n-i} | W) &= (1, i), \quad r(c_{n-i} | W) = (n - 1, i).
\end{align*}
\]

The only two different vertices \( x, x' \in V(P_3 \times C_n) \setminus W \) for which \( r(x | W) = r(x' | W) \) are \( c_0 \) and \( c_2 \), since \( r(c_0 | W) = r(c_2 | W) = (2, 2) \). However the vertices \( c_0, c_2 \) can be easily resolved for example by \( a_4 \). Thus the set \( \{a_0, a_2, a_4\} \) resolves all vertices in \( V(P_3 \times C_n) \setminus W \), which implies that \( W \) resolves the vertices in \( V(P_3 \times C_n) \setminus W \). Hence \( \text{dim}(P_3 \times C_n) = n \).

**Theorem 3.3** Let \( n \geq 7 \) be odd. Then \( \text{dim}(P_4 \times C_n) = \lfloor 2n/3 \rfloor \).

**Proof.** We present the distances between all pairs of vertices of \( P_4 \times C_n \). Let \( x = a_i \) or \( c_i \) and let \( y = b_i \) or \( e_i \), where \( i = 0, 1, 2, \ldots, n - 1 \). For \( p = \pm 2, \pm 4, \ldots, \pm t_1; s = \pm 1, \pm 3, \ldots, \pm t_2; t_1, t_2 \in \{(n - 3)/2, (n - 1)/2\} \) and \( \epsilon = \pm 1 \), we have

\[
\begin{align*}
&d(x, a_{i+p}) = d(x, c_{i+p}) = p, \quad d(x, a_{i-s}) = d(x, c_{i+s}) = n - s, \quad d(a_i, c_i) = 2, \\
&d(x, b_{i+p}) = d(x, e_{i+p}) = s, \quad d(x, b_{i-s}) = d(x, e_{i-s}) = n - p, \\
&d(x, b_i) = d(x, c_i) = n, \quad d(x, b_{i+\epsilon}) = d(x, c_{i+\epsilon}) = 1, \quad d(a_i, c_{i+\epsilon}) = 3, \\
&d(y, b_{i+p}) = d(y, e_{i+p}) = p, \quad d(y, b_{i-s}) = d(y, e_{i-s}) = n - s, \quad d(b_i, e_i) = 2.
\end{align*}
\]

First we prove that \( \text{dim}(P_4 \times C_n) \geq \lfloor 2n/3 \rfloor \). We show that we need to have at least \( \lfloor 2n/3 \rfloor \) vertices in a resolving set in order to resolve all pairs \( a_j, c_j \) and all pairs \( b_j, e_j \) for \( j = 0, 1, 2, \ldots, n - 1 \).

No vertex \( b_i \) resolves the vertices \( a_j, c_j \) for \( j = 0, 1, 2, \ldots, n - 1 \), and the vertices \( b_{j'}, e_{j'} \) for \( j' = 0, 1, 2, \ldots, n - 1, j' \neq j \), since \( d(b_i, a_j) = d(b_i, c_j) = d(b_{j'}, b_{j'}) \). Similarly, no vertex \( c_j \) can resolve the pairs \( b_j, e_j \) for \( j = 0, 1, 2, \ldots, n - 1 \) and the pairs \( a_j, c_j \) for \( j = 0, 1, 2, \ldots, n - 1, j' \neq j \). The vertex \( a_i \) cannot resolve the pairs \( a_{j'}, c_{j'} \) for \( j' = 0, 1, 2, \ldots, n - 1, j' \neq i \) and the pairs \( b_j, c_j \) for \( j = 0, 1, 2, \ldots, n - 1, j' \neq i - 1 \). Similarly, the vertex \( e_i \) resolves only the pair \( b_i, e_i \) and the pairs \( c_{i+\epsilon}, c_{i+\epsilon} \). It follows that among the pairs of vertices \( a_j, c_j \) and \( b_j, e_j \), where \( j = 0, 1, 2, \ldots, n - 1 \), any vertex of \( P_4 \times C_n \) can resolve at most 3 pairs. Since we have \( n \) pairs \( a_j, c_j \) and \( n \) pairs \( b_j, e_j \), in order to resolve \( 2n \) pairs of vertices, any resolving set

must contain at least 2n/3 vertices. The number of vertices in a resolving set is an integer, therefore \( \text{dim}(P_4 \times C_n) \geq [2n/3] \).

Now we find resolving sets, which consist of \([2n/3]\) vertices. Let us consider two cases.

Case 1: \( n \equiv 3 \text{ or } 5 \mod 6 \).

We show that \( W = \{a_i, e_i \mid i = 0, 3, 6, \ldots, k\} \), where \( k = n - 3 \) if \( n \equiv 3 \mod 6 \) and \( k = n - 2 \) if \( n \equiv 5 \mod 6 \), is a resolving set of \( P_4 \times C_n \). Let us present the representation of distances of all vertices of \( P_4 \times C_n \) with respect to \( W' = \{a_0, a_3\} \):

\[
\begin{align*}
r(a_i|W') &= r(c_i|W') = (i, n - i + 3), \quad i = 4, 6, \ldots, n - 1, \\
r(a_{n-i}|W') &= r(c_{n-i}|W') = (i, n - i - 3), \quad i = 2, 4, \ldots, n - 5, \\
r(b_i|W') &= r(e_i|W') = (i, n - i + 3), \quad i = 3, 5, \ldots, n, \\
r(b_{n-i}|W') &= r(e_{n-i}|W') = (i, n - i - 3), \quad i = 3, 5, \ldots, n - 6, \\
r(a_0|W') &= (0, n - 3), \quad r(c_0|W') = (2, n - 3), \\
r(a_1|W') &= r(c_1|W') = (n - 1, 2), \\
r(a_2|W') &= r(c_2|W') = (2, n - 1), \\
r(a_3|W') &= (n - 3, 0), \quad r(c_3|W') = (n - 3, 2), \\
r(b_1|W') &= (1, n - 2), \quad r(e_1|W') = (3, n - 2), \\
r(b_2|W') &= (n - 2, 1), \quad r(e_2|W') = (n - 2, 3), \\
r(b_4|W') &= (n - 4, 1), \quad r(e_4|W') = (n - 4, 3), \\
r(b_{n-1}|W') &= (1, n - 4), \quad r(e_{n-1}|W') = (3, n - 4).
\end{align*}
\]

It can be seen that except for some pairs \( a_j, c_j \) and \( b_j, e_j \), where \( 0 \leq j \leq n - 1 \), there are no other pairs of vertices having the same representation of distances with respect to \( W' \).

So it remains to show that all pairs \( a_j, c_j \) and all pairs \( b_j, e_j \) for \( j = 0, 1, 2, \ldots, n - 1 \), can be resolved by some vertices in \( W \). Let us present the distances of the vertices \( a_j, b_j, c_j, e_j \) for \( j = i - 1, i, i + 1 \) with respect to \( W_i = \{a_i, c_i\} \subset W \), where \( i \in \{0, 3, 6, \ldots, p\} \). Let \( \epsilon = \pm 1 \). Then

\[
\begin{align*}
r(a_{i+\epsilon}|W_i) &= (n - 1, 3), \quad r(c_{i+\epsilon}|W_i) = (n - 1, 1), \\
r(b_{i+\epsilon}|W_i) &= (1, n - 1), \quad r(e_{i+\epsilon}|W_i) = (3, n - 1), \\
r(a_i|W_i) &= (0, n), \quad r(c_i|W_i) = (2, n), \\
r(b_i|W_i) &= (n, 2), \quad r(e_i|W_i) = (0, n).
\end{align*}
\]

The set \( W_i \) resolves the pairs \( a_j, c_j \) and the pairs \( b_j, e_j \) for \( j = i - 1, i, i + 1 \), thus all pairs \( a_j, c_j \) and all pairs \( b_j, e_j \) for \( j = 0, 1, 2, \ldots, n - 1 \), are resolved by the vertices of \( W \), which means that \( W \) is a resolving set of \( P_4 \times C_n \). We have \( |W| = 2(k/3 + 1) \), which is equal to \( 2n/3 \) if \( n \equiv 3 \mod 6 \) or \( 2(n + 1)/3 \) if \( n \equiv 5 \mod 6 \). Thus \( |W| = [2n/3] \) and \( \text{dim}(P_4 \times C_n) = [2n/3] \).

Case 2: \( n \equiv 1 \mod 6 \).

We can write \( n = 6k + 1 \) where \( k \geq 1 \). We show that \( W = \{a_0\} \cup \{e_{i-4}, e_{i-3}, a_{i-1}, a_i \mid i = 6, 12, \ldots, 6k\} \), is a resolving set of \( P_4 \times C_n \). Let us present the representation of distances
of all vertices of $P_4 \times C_n$ with respect to $W' = \{a_0, e_2\} \subset W$. We have
\[
\begin{align*}
    r(a_i|W') &= r(c_i|W') = (i, n - i + 2), \quad i = 2, 4, \ldots, n - 1, \\
    r(a_{n-i}|W') &= r(c_{n-i}|W') = (i, n - i - 2), \quad i = 2, 4, \ldots, n - 5, \\
    r(b_i|W') &= r(e_i|W') = (i, n - i + 2), \quad i = 3, 5, \ldots, n, \\
    r(b_{n-i}|W') &= r(e_{n-i}|W') = (i, n - i - 2), \quad i = 3, 5, \ldots, n - 4, \\
    r(a_0|W') &= (0, n - 2), \quad r(e_0|W') = (2, n - 2), \\
    r(a_1|W') &= (n - 1, 3), \quad r(e_1|W') = (n - 1, 1), \\
    r(a_3|W') &= (n - 3, 3), \quad r(e_3|W') = (n - 3, 1), \\
    r(b_1|W') &= (1, n - 1), \quad r(e_1|W') = (3, n - 1), \\
    r(b_2|W') &= (n - 2, 2), \quad r(e_2|W') = (n - 2, 0), \\
    r(b_{n-1}|W') &= (1, n - 3), \quad r(e_{n-1}|W') = (3, n - 3).
\end{align*}
\]

Except for some pairs $a_j, c_j$ and $b_j, e_j$, where $0 \leq j \leq n - 1$, the only cases when vertices of $P_4 \times C_n$ have the same representation of distances with respect to $W'$ are the following ones:
\[
\begin{align*}
    r(a_1|W') &= r(a_{n-1}|W') = r(c_{n-1}|W') = (n - 1, 3), \\
    r(e_1|W') &= r(b_3|W') = r(e_3|W') = (3, n - 1).
\end{align*}
\]

However, it is easy to find a vertex in $W$, for example $a_5$, which can solve this problem. We have
\[
\begin{align*}
    d(a_5, a_1) &= 4, \quad d(a_5, a_{n-1}) = d(a_5, c_{n-1}) = 6 \quad \text{and} \\
    d(a_5, e_1) &= n - 4, \quad d(a_5, b_3) = d(a_5, e_3) = n - 2.
\end{align*}
\]

Finally we prove that all pairs $a_j, c_j$ and all pairs $b_j, e_j$ for $j = 0, 1, 2, \ldots, n - 1$, can be resolved by some vertices in $W$. Any vertex $e_{l}, 0 \leq l \leq n - 1$ resolves the pairs $b_l, e_l$ and $a_{l+\epsilon}, c_{l+\epsilon}$ for $\epsilon = \pm 1$, since $d(e_l, b_l) = 2$ and $d(e_l, a_{l+\epsilon}) = 3 = d(e_l, c_{l+\epsilon})$. Similarly, any vertex $a_l$ resolves the pairs $a_l, e_l$ and $b_l, e_{l+\epsilon}$, where $\epsilon = \pm 1$. This implies that the set $W_i = \{e_{i-4}, e_{i-3}, a_{i-1}, a_i\} \subset W$, where $i \in \{6, 12, \ldots, 6k\}$, resolves the pairs $a_j, c_j$ for $j = i - 5, i - 4, \ldots, i$, and the pairs $b_j, e_j$ for $j = i - 4, i - 3, \ldots, i + 1$. The only pairs not resolved by $W_6 \cup W_{12} \cup \cdots \cup W_{6k}$ are $a_0, c_0$ and $b_1, e_1$. These pairs are resolved by $a_0$, hence $\text{dim}(P_4 \times C_n) = |W| = 1 + 4k = (2n + 1)/3 = \lceil 2n/3 \rceil$. The proof is complete. ■

4. Conclusion

Finding the metric dimension of a graph is an NP-hard problem. In this paper we considered the connectivity and the metric dimension of the graphs $C_m \times C_n$, $P_m \times C_n$ and $P_m \times P_n$ for $m, n \geq 2$.

By Lemma 2.3, the graph $P_m \times P_n$ has two components, thus $\text{dim}(P_m \times P_n) = \infty$ for any $m, n \geq 2$.

Theorems 3.2 and 3.3 give the values of $\text{dim}(P_3 \times C_n)$ if $n \geq 5$ is odd, and $\text{dim}(P_4 \times C_n)$ if $n \geq 7$ is odd. Let us note that $\text{dim}(P_3 \times C_3) = 5$ where $\{a_0, c_0, a_1, c_1, a_2\}$ is the resolving set of $P_3 \times C_3$, and $\text{dim}(P_4 \times C_3) = \text{dim}(P_4 \times C_5) = 4$ where $\{a_0, c_0, a_2, e_2\}$ is the resolving set of $P_4 \times C_3$ and $P_4 \times C_5$. By Lemma 2.2, we have $\text{dim}(P_m \times C_n) = \infty$ if $n$ is even. It
would be interesting to know the values of the metric dimension of $P_m \times C_n$ for $m \geq 5$. Therefore, let us formulate the following problem.

**Problem 4.1** Determine $\dim(P_m \times C_n)$ for $m \geq 5$ and odd $n \geq 3$.

Since by Theorem 3.1, $\dim(C_3 \times C_n) = \lceil 2n/3 \rceil$ for any $n \geq 9$ and from Lemma 2.1, we get $\dim(C_m \times C_n) = \infty$ if both $m$ and $n$ are even, we can define Problem 4.2.

**Problem 4.2** Determine $\dim(C_m \times C_n)$ for $m \geq 4$ and odd $n \geq 5$.

These challenging problems remain open for future research.

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**References**


