LIST COLORING OF COMPLETE MULTIPARTITE GRAPHS

Tomáš Vetrík
School of Mathematical Sciences
University of KwaZulu-Natal
Durban, South Africa
e-mail: tomas.vetrik@gmail.com

Abstract

The choice number of a graph $G$ is the smallest integer $k$ such that for every assignment of a list $L(v)$ of $k$ colors to each vertex $v$ of $G$, there is a proper coloring of $G$ that assigns to each vertex $v$ a color from $L(v)$. We present upper and lower bounds on the choice number of complete multipartite graphs with partite classes of equal sizes and complete $r$-partite graphs with $r - 1$ partite classes of order two.

Keywords: list coloring, choice number, complete multipartite graph.

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1. Introduction

All graphs considered here are finite, undirected, without loops and multiple edges. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A list assignment to the vertices of a graph $G$ is the assignment of a list $L(v)$ of colors $C$ to every vertex $v \in V(G)$. A $k$-list assignment is a list assignment such that $|L(v)| \geq k$ for every vertex $v$. An $L$-coloring of $G$ is a function $f : V(G) \rightarrow C$ such that $f(v) \in L(v)$ for all $v \in V(G)$ and $f(v) \neq f(w)$ for each edge $vw \in E(G)$. If $G$ has an $L$-coloring, then $G$ is said to be $L$-colorable. If for any $k$-list assignment $L$ there exists an $L$-coloring, then $G$ is $k$-choosable. The choice number $Ch(G)$ of a graph $G$ is the minimum integer $k$ such that $G$ is $k$-choosable.

The study of choice numbers of graphs was initiated by Vizing [7] and by Erdős, Rubin and Taylor [3]. For a survey about the list coloring problem we refer to [6] and [8]. In this paper we focus on the choice numbers of complete multipartite graphs.
2. Complete multipartite graphs with partite classes of different sizes

Let $K_{n_1,n_2,\ldots,n_r}$ be the complete $r$-partite graph with the partite classes of order $n_1,n_2,\ldots,n_r$. A well-known result of Erdős, Rubin and Taylor [3] says that the choice number of the complete $r$-partite graph $K_{2,2,\ldots,2}$ is $r$. Gravier and Maffray [4] proved that also $Ch(K_{3,3,2,\ldots,2}) = r$ for $r \geq 3$. Enomoto et. al. [2] showed that $Ch(K_{5,2,\ldots,2}) = r + 1$ and the choice number of the complete $r$-partite graph $K_{4,2,\ldots,2}$ is equal to $r$ if $r$ is odd, and $r + 1$ if $r$ is even.

Motivated by these results we study the value $Ch(K_{n,2,\ldots,2})$ for any positive integer $n$. In the proof of Theorem 1 we write $L(S)$ for the union $\bigcup_{v \in S} L(v)$ where $S \subseteq V(G)$. If $C$ is a set of colors, then $L \setminus C$ denotes the list assignment obtained from $L$ by removing the colors in $C$ from each $L(v)$ where $v \in V(G)$. First, we show that the graph $K_{(t+2)(t+3)/2,2,\ldots,2}$ is $(r+t)$-choosable.

**Theorem 1.** Let $t$ be a positive integer and let $G$ be a complete $r$-partite graph with one partite class of order $(t+2)(t+3)/2$ and $r - 1$ partite classes of order two. Then $Ch(G) \leq r + t$.

**Proof.** Let $V_1$ be the partite class of $G$ of order $(t+2)(t+3)/2$ and let $V_i = \{v_i,w_i\}, 2 \leq i \leq r$, be the partite classes of order two. Let $L_1$ be any $(r+t)$-list assignment to the vertices of $G$. We prove that $G$ is $L_1$-colorable. We distinguish three cases:

CASE 1: $t \geq r - 1$.

We can color the vertices of $V_2, V_3, \ldots, V_r$ with $2r - 2$ different colors. Since $|L_1(v)| \geq 2r - 1$ for every vertex $v \in V_1$, we can color the vertices of $V_1$ as well.

CASE 2: There exists a color $c \in L_1(v_i) \cap L_1(w_i)$ for some $i \in \{2, 3, \ldots, r\}$.

It is easy to show by induction on $r$ that $G$ is $L_1$-colorable. The step $r = 1$ is trivial. For the induction step, assign $c$ to both $v_i$ and $w_i$, and remove $c$ from the lists of the remaining vertices. By the induction hypothesis, the remaining vertices can be colored with colors from the revised lists.

CASE 3: $t \leq r - 2$ and $L_1(v_i) \cap L_1(w_i) = \emptyset$ for every $i \in \{2, 3, \ldots, r\}$.

We prove by contradiction that $G$ is $L_1$-colorable. Assume that $G$ is not $L_1$-colorable. Let $L$ be an $(r+t)$-list assignment such that $G$ is not $L$-colorable. Let $X_j, j = 1, 2, \ldots, t$, be the largest subset of $V_1 \setminus (\bigcup_{l=1}^{t-1} X_l)$ with
\[ \bigcap_{v \in X_j} L(v) \neq \emptyset. \] Set \[ |X_j| = x_j \] and choose a color \[ c_j \in \bigcap_{v \in X_j} L(v). \] Define \[ L' = L \setminus \{c_1, c_2, \ldots, c_t\} \] and \[ G' = G \setminus (\bigcup_{i=1}^t X_i). \] Note that \[ |L'(v)| = r + t \] for each \[ v \in V(G') \cap V_1 \] and \[ |L'(w_i)|, |L'(w_j)| \geq r \] for any \[ i \in \{2, 3, \ldots, r\}. \] Since \[ G \] is not \( L \)-colorable, \( G' \) is not \( L' \)-colorable. It follows that there exists a set of vertices \[ T \subseteq V(G') \] such that \[ |L'(T)| < |T|, \] i.e., \( L' \) does not satisfy Hall’s condition. Let \( S \) denote a maximal subset of \( V(G') \) such that \[ |L'(S)| < |S|. \] We consider two subcases:

CASE 3a: \[ |S \cap V_i| \leq 1 \] for every \( i \in \{2, 3, \ldots, r\}. \)

Since \[ |L'(v_i)|, |L'(w_i)| \geq r \] and \[ |S \cap V_i| \leq r - 1 \], \( S \cap V_i \) can be colored from the list \( L' \). Further, \[ |L'(v)| = r + t \] for \( v \in S \cap V_i \), therefore we can also color the vertices in \( S \cap V_i \). Let \[ L'' = L' \setminus L'(S). \] We show that \( G' \setminus S \) is \( L'' \)-colorable. If \( G' \setminus S \) is not \( L'' \)-colorable, we have a nonempty subset \( S' \subseteq V(G') \setminus S \) with \[ |L''(S')| < |S'|. \] Then \[ |L'(S \cup S')| = |L'(S)| + |L''(S')| < |S| + |S'|, \] which contradicts the maximality of \( S \).

CASE 3b: Both \( v_i, w_i \in S \) for some \( i \in \{2, 3, \ldots, r\}. \)

Then \[ |S| > |L'(S)| \geq |L'(v_i)| + |L'(w_i)| \geq 2(r + t) - t. \] Let \[ |S| = 2r + t + 1 + \epsilon \] where \( \epsilon \geq 0 \). Clearly, \[ |L'(S)| \leq 2r + t + \epsilon. \] Let \( S_1 = S \cap V_i \). We have \[ |S_1| \geq |S| - (2r - 2) = t + 3 + \epsilon. \] By the maximality of \( X_t \), every color in \( L'(S) \) appears in the lists of at most \( x_t \) vertices of \( S_1 \). It means that

\[ (r + t)|S_1| = \sum_{v \in S_1} |L'(v)| \leq x_t|L'(S)|. \]  

It is evident that \[ \sum_{t=1}^{t} x_t + |S_1| \leq |V_t| = (t + 2)(t + 3)/2. \] Hence, \[ t x_t + |S_1| \leq (t + 2)(t + 3)/2, \] or equivalently

\[ x_t \leq \left[ (t + 2)(t + 3)/2 - |S_1| \right] / t \]  

By (1) and (2), we have \( (r + t)|S_1| \leq [(t + 2)(t + 3)/2 - |S_1|]|L'(S)|/t. \) Since \[ |S_1| \geq t + 3 + \epsilon \] and \[ |L'(S)| \leq 2r + t + \epsilon, \] we have \( (r + t)(t + 3 + \epsilon) \leq [(t + 2)(t + 3)/2 - (t + 3 + \epsilon)](2r + t + \epsilon)/t \) which yields \( \frac{t^2}{2} + (3 + \epsilon)\frac{t^2}{2} + (r - \frac{1}{2})\epsilon t + (2r + \epsilon)\epsilon \leq 0, \) a contradiction. This finishes the proof. \( \square \)

If \( t = 1 \), then \( Ch(K_{5,2,\ldots,2}) \leq r + 1. \) This bound also comes from the result \( Ch(K_{3,3,2,\ldots,2}) = r \) of Gravier and Maffray [4], because the complete \( r \)-partite graph \( K_{5,2,\ldots,2} \) is a subgraph of the complete \( (r + 1) \)-partite graph \( K_{3,3,2,\ldots,2}. \) Since the choice number of the complete \( r \)-partite graph \( K_{5,2,\ldots,2} \) is equal to \( r + 1 \), it is clear that \( Ch(K_{5,2,\ldots,2}) = r + 1 \) as well.
Now we present a lower bound on the choice number of complete \( r \)-partite graphs with \( r - 1 \) partite classes of order at most two.

**Theorem 2.** Let \( s, r, t \) be integers such that \( 0 \leq s < r \) and \( t > 0 \). Let \( G \) be a complete \( r \)-partite graph consisting of one partite class of order \( \binom{2t+s}{t} \), \( r-s-1 \) partite classes of order two, and \( s \) partite classes of order one. Then

\[
Ch(G) > \left\lceil \frac{r+t-1}{2t+s} \frac{1}{2t+s} \right\rceil (2t+s).
\]

**Proof.** Let \( n = \binom{2t+s}{t} \) and \( m = \frac{r+t-1}{2t+s} \). Let \( G \) be a complete \( r \)-partite graph with the partite classes \( V_i, V_i = \{v_i, w_i\}, V_j = \{v_j\} \), where \( |V_i| = n; i = 2, 3, \ldots, r-s \) and \( j = r-s+1, r-s+2, \ldots, r \). Let \( A_1, A_2, \ldots, A_{2t+s}, B_1, B_2, \ldots, B_{2t+s} \) be disjoint color sets of order \( m \) such that \( \bigcup_{i=1}^{2t+s} A_i = A, \bigcup_{i=1}^{2t+s} B_i = B \). We define a list assignment \( L \) to \( V(G) \) by the following way:

\[
L(v_i) = A, j = 2, 3, \ldots, r,
L(w_i) = B, i = 2, 3, \ldots, r-s.
\]

The lists of colors given to the vertices of \( V_1 \) consist of \( 2t+s \) different sets

\[
A_{x_1}, A_{x_2}, \ldots, A_{x_t+s}, B_{w_1}, B_{w_2}, \ldots, B_{w_t},
\]

where \( x_1, x_2, \ldots, x_{t+s}, y_1, y_2, \ldots, y_t \in \{1, 2, 2t+s\} \). Since the number of vertices in \( V_1 \) is \( n = \binom{2t+s}{t} \binom{2t+s}{t} \), we are able to assign to any two vertices in \( V_1 \) different lists.

We show by contradiction that \( G \) cannot be colored from the list \( L \). Suppose that \( G \) can be colored from \( L \). We use \( r-1 \) different colors of \( A \) to color the vertices \( v_2, v_3, \ldots, v_r \) and \( r-s-1 \) different colors of \( B \) to color \( w_2, w_3, \ldots, w_{r-s} \). Since \( |A| = |B| = |m| (2t+s) \leq r+t-1 \), the number of colors in \( A \) (in \( B \)) not used to color \( V_2, V_3, \ldots, V_r \) is at most \( t \) (at most \( t+s \)). It follows that there are at most \( 2t+s \) sets

\[
A_{x_1'}, A_{x_2'}, \ldots, A_{x_t'}, B_{y_1'}, B_{y_2'}, \ldots, B_{y_{t+s}'} \]

where \( x_1', x_2', \ldots, x_t', y_1', y_2', \ldots, y_{t+s}' \in \{1, 2, \ldots, 2t+s\} \) containing colors that were not employed in coloring \( V_2, V_3, \ldots, V_r \). Try to color \( V_1 \) with these colors. According to the assignment of color sets to the vertices of \( V_1 \), there exists a vertex \( v \in V_1 \) having none of the sets \( A_{x_1'}, A_{x_2'}, \ldots, A_{x_t'}, B_{y_1'}, B_{y_2'}, \ldots, B_{y_{t+s}'} \) in its list, a contradiction. Hence, \( G \) is not \( L \)-colorable. \( \square \)

Note that we get the bound \( Ch(K_{(\binom{2t}{t})^2, \ldots, 2}) \geq r+t \) if \( s = 0 \) and \( r = pt+1 \) for some odd integer \( p \).

3. Complete multipartite graphs with partite classes of equal sizes

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Let $K_{n,r}$ denote the complete multipartite graph with $r$ partite classes of order $n$. The problem is to determine the value of the choice number $Ch(K_{n,r})$. If $n = 1$, then $K_{n,r}$ is a clique on $r$ vertices and hence, obviously, $Ch(K_{1,r}) = r$. In the previous section we mentioned that $Ch(K_{2,r}) = r$ as well. Alon [1] established the general bounds $c_1 r \log n \leq Ch(K_{n,r}) \leq c_2 r \log n$ for every $r, n \geq 2$, where $c_1, c_2$ are two positive constants. Later, Kierstead [5] solved the problem in the case $n = 3$. He showed that $Ch(K_{3,r}) = \left\lceil \frac{4r-1}{3} \right\rceil$. Yang [9] studied the value of $Ch(K_{4,r})$ and obtained the bounds $\left\lceil \frac{2r}{3} \right\rceil \leq Ch(K_{4,r}) \leq \left\lceil \frac{2r}{3} \right\rceil$. We present results giving exact bounds on $Ch(K_{n,r})$ for large $n$. In the proof of Theorem 3 we use the following lemma proved in [5].

**Lemma 1.** A graph $G$ is $k$-choosable if $G$ is $L$-colorable for every $k$-list assignment $L$ such that $|\bigcup_{v \in V(G)} L(v)| < |V(G)|$.

Let us derive an upper bound on the choice number of complete multipartite graphs with partite classes of equal sizes.

**Theorem 3.** Let $0 < \alpha < n$ and let $x_j = \lfloor (\alpha - \frac{\alpha}{n} \sum_{i=1}^{j-1} x_i) \rfloor + 1$, $j = 1, 2, \ldots, \lfloor \alpha \rfloor$. If $n \leq \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$, then $Ch(K_{n,r}) \leq \lfloor \alpha r \rfloor$.

**Proof.** Let $V_i$, $i = 1, 2, \ldots, r$, be the $i$-th partite class of $K_{n,r}$. We prove the result by induction on $r$. The case $r = 1$ is trivial. For the induction step consider an $[\alpha r]$-list assignment $L$ to the vertices of $K_{n,r}$. We prove that if $n \leq \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$, then any partite class $V_i$ can be colored with at most $\lfloor \alpha \rfloor$ colors.

Assume that $n = \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$. In this paragraph we show by induction on $j$ ($j = 1, 2, \ldots, \lfloor \alpha \rfloor$), that there exists a color $c_j$ which can be used for coloring $x_j$ vertices of $V_i$ that have not been colored by $c_1, c_2, \ldots, c_{j-1}$ yet. Note that $c_j, c_j'$, where $l, l' \in \{1, 2, \ldots, \lfloor \alpha \rfloor\}$, $l \neq l'$, do not have to be different. If $j = 1$, we have $x_1 = \lfloor \alpha \rfloor + 1$. Since $\sum_{v \in V_1} |L(v)| = [\alpha r]n$ and by Lemma 1, $|\bigcup_{v \in V(K_{n,r})} L(v)| < rn$, there exists a color $c_1$ which appears in the lists of at least $\lfloor \alpha \rfloor + 1$ vertices of $V_i$. Color these vertices with $c_1$. Suppose $j \geq 2$. We can color $\sum_{i=1}^{j-1} x_i$ vertices with $c_1, c_2, \ldots, c_{j-1}$. The sum of the numbers of colors in the lists of the remaining $n - \sum_{i=1}^{j-1} x_i$ vertices of $V_i$ is $(n - \sum_{i=1}^{j-1} x_i) [\alpha r]$. Since $|\bigcup_{v \in V_i} L(v)| < rn$, there is a color $c_j$ that appears in the lists of $|\bigcup_{v \in V_i} L(v)| < rn$ at least $\lfloor \alpha \rfloor$ times. Hence, we can color these vertices with $c_j$. It follows that it is possible to color $n = \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$ vertices of $V_i$ with at most $\lfloor \alpha \rfloor$ different colors.
Clearly, if \( n < \sum_{i=1}^{n} |a_i| \), all the vertices of \( V_i \) can be colored with at most \( |a_i| \) colors too. Let us remove the colors that were employed in coloring \( V_i \) from the lists given to the vertices in \( V(K_{n,r}) \). We have at least \( |\alpha r| - |\alpha| \geq |\alpha(r-1)| \), by applying the induction hypothesis, \( r-1 \) partite classes can be colored with \( |\alpha(r-1)| \) colors, i.e., there exists a proper coloring of the vertices in \( V(K_{n,r}) \) with colors from the revised lists. \( \square \)

Unfortunately, the result presented in Theorem 3 can not be bounded from above by \( cn \log n \), where \( c \) is a constant. Theorem 3, for example, yields the upper bounds \( Ch(K_{5, r}) \leq \lceil \frac{3}{2} r \rceil \), \( Ch(K_{15, r}) \leq 5r \), \( Ch(K_{40, r}) \leq 10r \), \( Ch(K_{75, r}) \leq 15r \) and \( Ch(K_{121, r}) \leq 20r \). One can check that \( 10r \approx 6.24r \log 40, 15r \approx 8r \log 75 \) and \( 20r \approx 9.6r \log 121 \).

The following result gives a lower bound on \( Ch(K_{n,r}) \).

**Theorem 4.** Let \( x, t, r, n \) be integers such that \( x, t, r \geq 2 \), \( x \geq t \) and \( n = \left\lceil \frac{x}{x-t+1} \right\rceil \). Then \( Ch(K_{n,r}) > (x-t+1)\left\lceil \frac{r-1}{x} \right\rceil \).

**Proof.** Let \( x, t, r \geq 2 \), \( x \geq t \), \( n = \left\lceil \frac{x}{x-t+1} \right\rceil \) and let \( k = (x-t+1)\left\lceil \frac{r-1}{x} \right\rceil \). We show that there exists a \( k \)-list assignment \( L \) of \( K_{n,r} \) such that \( K_{n,r} \) is not \( L \)-colorable. Let \( V_i \), \( i = 1, 2, \ldots, r \), be the \( i \)-th partite class of \( K_{n,r} \). Let \( A_1, A_2, \ldots, A_x \) be a family of disjoint color sets such that \( |A_j| = |A_i| \) or \( |A_j| = |A_i| + 1, j = 2, 3, \ldots, x \), and \( |\bigcup_{j=1}^{x} A_j| = tr - 1 \). Obviously, \( |A_j| \geq \left\lceil \frac{r-1}{x} \right\rceil \) for any \( j \in \{1, 2, \ldots, x\} \).

Define a list assignment \( L \) as follows: Let the lists given to the \( n \) vertices of every partite class \( V_i \) consist of \( x-t+1 \) different sets \( A_{y_1}, A_{y_2}, \ldots, A_{y_{x-t+1}} \), \( y_1, y_2, \ldots, y_{x-t+1} \in \{1, 2, \ldots, x\} \), where any two vertices in the same part have different lists. Note that \( |L(v)| \geq (x-t+1)\left\lceil \frac{r-1}{x} \right\rceil \) for each vertex \( v \in V(K_{n,r}) \). Then for any partite class \( V_i \) and any \( t-1 \) colors \( a_j \in A_{y_j} \), \( j = 1, 2, \ldots, x-t+1 \) there is a vertex \( v \in V_i \) having none of the sets \( A_{y_j} \) in its list. Therefore, in any coloring from these lists, we must use at least \( t \) colors on each partite class. Since the number of colors in \( \bigcup_{j=1}^{x} A_j \) is less than \( tr \), \( K_{n,r} \) is not \( L \)-colorable. \( \square \)

Theorem 4 says that if, for instance \( t = 2 \), then \( n = x \) and \( Ch(K_{n,r}) > (n-1)\left\lceil \frac{r-1}{n} \right\rceil \). In particular, for \( n = 5 \) we have \( Ch(K_{5,r}) > 4\left\lceil \frac{2r-1}{5} \right\rceil \). If \( t = 3 \), then \( Ch(K_{n,r}) > (x-2)\left\lceil \frac{3r-1}{x} \right\rceil \). For example, in the case \( x = 6 \) we get \( Ch(K_{15,r}) > 4\left\lceil \frac{3r-1}{6} \right\rceil = 4\left\lceil \frac{r-1}{2} \right\rceil \).

Finally, we present a corollary of Theorem 4 which yields a lower bound.
in the form $cr \log n$.

**Corollary 1.** Let $r \geq 2$ and $n = \left(\frac{x}{|x/2|}\right)$ where $x \geq 5$. Then $Ch(K_{n^r}) > \left[\frac{x}{2}\right]\left\lceil\log_{2.1^n}\right\rceil$.

**Proof.** For $x, t, r \geq 2$, $x \geq t$ and $n = \left(\frac{x}{|x-t+1|}\right)$, we have $Ch(K_{n^r}) > (x - t + 1)\left[\frac{x-1}{x}\right]$. Let $t = \left[\frac{x}{2}\right] + 1$. Then $Ch(K_{n^r}) > \left[\frac{x}{2}\right]\left\lceil\frac{|x/2|+r-1}{x}\right\rceil > \left[\frac{x}{2}\right]\left[\frac{x}{2}\right]$. It is well-known that $\frac{x^r}{e^r} \leq x! \leq \frac{(x+1)^{x+1}}{e^{x+1}}$ for any $x$. For $x \geq 5$, the following inequalities also hold: $\frac{2x^x}{e^x} < x! < \frac{6x^{x+1}}{5e^{x+1}}$. Then $n = \frac{x^t}{|x/2||x/2|} < \frac{6x^{x+1}/(5e^x)}{4|x/2|^{|x/2|}/|x/2|^{|x/2|}/e^{x+1}} \leq \frac{3x^{x+1}}{10|x/2|^{|x/2|}/e^2} \leq \frac{3x^{x+1}/x^{x+1}/e^2}{10(x-1)^{x+1}}$. Since $x^2 \cdot 2^x < 7.6(2.1)^x$ for any $x$ (note that $7.5(2.1)^x < x2^x$ for $19 \leq x \leq 22$) and $(\frac{x}{x-1})^x < 3.1$ for any $x \geq 5$, we have $n < \frac{9.068(2.1)^x}{x^{x+1}/e^2} < (2.1)^x$. Consequently, $\log_{2.1^n} x < x$, hence $Ch(K_{n^r}) > \left[\frac{x}{2}\right]\left\lceil\log_{2.1^n}\right\rceil$ for any $n = \left(\frac{x}{|x/2|}\right)$ where $x \geq 5$. □

**References**


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