Large Cayley digraphs of given degree and diameter

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Abstract
We construct a family of Cayley digraphs of degree $d$, diameter $k$ and order $k\lfloor d/2 \rfloor^k$ for any $d \geq 4$ and $k \geq 3$. We also present a collection of bipartite Cayley digraphs of order at least $(k - 1)\lfloor d/2 \rfloor^{k-1}$ for any degree $d \geq 4$ and diameter $k \geq 4$. For sufficiently large $d$ and $k$, our digraphs are the largest known Cayley digraphs of degree $d$ and diameter $k$, where $k \neq d - 1$ or $d$, and our bipartite digraphs are the largest known bipartite Cayley digraphs.

Keywords: Cayley digraph, bipartite digraph, degree, diameter

1. Introduction and results

The directed degree-diameter problem is to determine the largest order of directed graphs (digraphs) of given maximum (out- and in-) degree and diameter. For details about the history and the current state of the problem we refer to [7]. In this note we focus on Cayley digraphs and bipartite Cayley digraphs of given degree and diameter.

Let $C_{d,k}$ be the largest order of a Cayley digraph of diameter $k$ and degree $d$ (the in-degree and out-degree of every vertex equals $d$). Clearly, the number of vertices in a digraph of maximum degree $d$ and diameter $k$ can not exceed the Moore bound $1 + d + d^2 + \ldots + d^k$. There is no general upper bound on $C_{d,k}$ better than the Moore bound, but it is known that it is not possible to construct Cayley digraphs of Abelian groups of order greater than $\binom{k+d}{d}$, see [2]. This bound differs from the Moore bound rather dramatically.

Cayley digraphs of order equal to the Moore bound exist only in the trivial cases when $d = 1$ or $k = 1$. The largest Cayley digraphs of degree 1 are the directed cycles of length $k + 1$ and the largest digraphs of diameter 1 are the complete digraphs of order $d + 1$. In general, it is difficult to determine the exact values of $C_{d,k}$. The largest known Cayley digraphs of given degree $d$
and diameter $k$ for small $d$ and $k$ were studied for example by Hafner [5]. We present a family of Cayley digraphs of degree $d \geq 4$, diameter $k \geq 3$ and order $k\lfloor d/2 \rfloor^k$.

**Theorem 1.** Let $d \geq 4$ and $k \geq 3$. Then $C_{d,k} \geq k\lfloor d/2 \rfloor^k$.

We compare Theorem 1 with other results on large Cayley digraphs of given degree and diameter. Garcia and Peyrat [4], and Macbeth et. al. [6] constructed large undirected Cayley graphs. After replacing every edge by a pair of oppositely directed edges, we get $C_{d,k} \geq d^k / (2k)!$ for sufficiently large $d$ and $k \geq d$ (see [4]), and $C_{d,k} \geq k((d-3)/3)^k$ for any degree $d \geq 5$ and diameter $k \geq 3$ (see [6]). It is evident that the number of vertices in our Cayley digraphs is greater than the bounds which come from [4] and [6].

The upper bound on the number of vertices in Abelian Cayley digraphs can be expressed in the form $k^d / d! + O(k^{d-1})$ for fixed $d$ and $k \to \infty$, and $d^k / k! + O(d^{k-1})$ for fixed $k$ and $d \to \infty$. For large $d$ and $k$, the order of our digraphs is clearly greater than $d^k / k! + O(d^{k-1})$. For fixed $d \geq 4$ and $k \to \infty$, we have $[d/2]^k > k^d$ which yields $k\lfloor d/2 \rfloor^k > d^k / d! + O(d^{k-1})$ for large $d$ and $k \to \infty$. Hence, if $d$ and $k$ are sufficiently large, it is not possible to construct Abelian Cayley digraphs of degree $d$ and diameter $k$ larger than our digraphs. The last result which yields record bounds on $C_{d,k}$ for large $d$ and $k$ seems to be a family of vertex-transitive digraphs of Faber, Moore and Chen [3]. The order of Faber-Moore-Chen digraphs is $(d+1)!/(d-k+1)!$ where $2 \leq k \leq d$, and one can see that they are larger than our digraphs. However, Ždimalová and Staneková [9] proved that the Faber-Moore-Chen digraphs are Cayley digraphs if and only if

(i) $(k, d) = (2, q - 1), (3, q), (4, 10), (5, 11)$, where $q$ is a prime power,
(ii) $k = d$ or $k = d - 1$.

For $k = 2$ the digraphs reduce to the Kautz digraphs. We have $C_{d,2} = d^2 + d$ for $d = q - 1$, where $q \geq 3$ is a prime power, and $C_{d,3} \geq d^3 - d$ for diameter 3 and degree $d \geq 3$, where $d$ is a prime power. Since in the majority of cases, the Faber-Moore-Chen digraphs are not Cayley, we can conclude that our digraphs are the largest known Cayley digraphs of degree $d$ and diameter $k$ for sufficiently large $d$ and $k$, where $k \neq d - 1$ or $d$. Let us also note that the Faber-Moore-Chen digraphs were constructed for $k \leq d$.

We also present a collection of Cayley digraphs of diameter 2.

**Assertion 1.** There exists Cayley digraphs of diameter 2, degree $d$ and order
at least $2(2d - 1)^2/9$ for any $d \geq 3$.

Our digraphs of diameter 2 are smaller than the Kautz digraphs of diameter 2, however by [9], the Kautz digraphs of degree $d$ and diameter 2 are Cayley if and only if $d = q - 1$, where $q$ is a prime power. Moreover, we believe that if one is able to generalize Assertion 1 for $k \geq 3$, we could get new interesting bounds on $C_{d,k}$ for large diameters in future.

Now let us study bipartite Cayley digraphs. Let $BC_{d,k}$ denote the largest possible number of vertices in a bipartite Cayley digraph of degree $d$ and diameter $k$. Aider [1] showed that the number of vertices in a bipartite digraph can not exceed the bound $2(1 + d^2 + \ldots + d^{k-1})$ if $k$ is odd, and $2d(1 + d^2 + \ldots + d^{k-2})$ if $k$ is even.

Exact values of $BC_{d,k}$ are available only in rare cases. For instance, it is easy to see that $BC_{1,k} = k + 1$ for an odd $k$ and $BC_{d,2} = 2d$ for $d \geq 2$. There is only one work, which states lower bounds on $BC_{d,k}$ for large $d$ and $k$. The bound $BC_{d,k} \geq 2(k - 1)((d - 4)/3)^{k-1}$ for $d \geq 6$ and even $k \geq 4$ follows from [8]. We construct bipartite Cayley digraphs for any degree $d \geq 4$ and diameter $k \geq 3$.

**Theorem 2.**

(i) Let $k \geq 4$ be even and $d \geq 4$. Then $BC_{d,k} \geq 2(k - 1)[d/2]^{k-1}$.

(ii) Let $k \geq 5$ be odd and $d \geq 4$. Then $BC_{d,k} \geq (k - 1)[d/2]^{k-1}$.

(iii) Let $d \geq 2$. Then $BC_{d,3} \geq 2d^2$.

Note that our digraphs of diameter 3 miss the upper bound of Aider [1] only by 2. To the best of our knowledge, there is no construction of bipartite Cayley digraphs of order greater than the order of our bipartite digraphs.

2. Proofs

In proofs we will use a group $H$ of order $m \geq 2$ with unit element $\iota$. We denote by $H^t$ the product $H \times H \times \ldots \times H$, where $H$ appears $t$ times. Consider the automorphism $\alpha$ of the group $H^t$ which shifts coordinates by one to the right, that is, $\alpha(x_1, x_2, \ldots, x_t) = (x_t, x_1, x_2, \ldots, x_{t-1})$. The cyclic group of order $p$ (where $p = t$ or $2t$) will be denoted by $Z_p$ and the semidirect product $H^t \rtimes Z_p$ will be denoted by $G$. We define multiplication in $G$ as follows: $(x, y)(x', y') = (x\alpha^y(x'), y + y')$, where $x, x' \in H^t$, $y, y' \in Z_p$, and $\alpha^y$ is the composition of $\alpha$ with itself $y$ times. Note that if $p = t$, then $G$ is the regular wreath product. Elements of $G$ will be written in the form


$$(x_1, x_2, \ldots, x_t; y),$$ where $x_1, x_2, \ldots, x_t \in H$ and $y \in Z_p$.

**Proof of Theorem 1.** Let $t = p = k \geq 3$. Let $a_g = (g, t, \ldots, t; 1)$, $b_h = (t, h, t, \ldots, t; 2)$ for any $g, h \in H$; $a_g, b_h \in G$. Let $X = \{a_g, b_h \mid \text{for all } g, h \in H\}$. The Cayley digraph $C(G, X)$ is of degree $d = |X| = 2m$, $m \geq 2$ and order $|G| = km^k = k(d/2)^k$.

We prove that the diameter of $C(G, X)$ is at most $k$, which is equivalent to showing that each element of $G$ can be obtained as a product of at most $k$ elements of $X$. Let $0 \leq y \leq k - 1$. Then

$$
(x_1, x_2, \ldots, x_k; y) = \left( \prod_{i=1}^{(y+1)/2} b_{x_{2i}} \right) \left( \prod_{j=1}^{k-1-y} a_{x_{j+y+1}} \right) a_{x_1} \left( \prod_{\ell=1}^{(y-1)/2} b_{x_{2\ell+1}} \right)
$$

if $y$ is odd, and

$$
(x_1, x_2, \ldots, x_k; y) = a_{x_1} \left( \prod_{i=1}^{y/2} b_{x_{2i+1}} \right) \left( \prod_{j=1}^{k-1-y} a_{x_{j+y+1}} \right) \left( \prod_{\ell=1}^{y/2} b_{x_{2\ell}} \right)
$$

if $y$ is even, where $x_1, x_2, \ldots, x_k$ are any elements of $H$.

Note that none of the elements $(x_1, x_2, \ldots, x_k; y)$ where all $x_i \neq t$, $i = 1, 2, \ldots, k$, can be obtained as a product of fewer than $k$ elements of $X$, therefore the diameter of $C(G, X)$ is exactly $k$.

By adding to the generating set $X$ a new element of $G$, we obtain a Cayley digraph of degree $d = 2m + 1$, diameter at most $k$ and order $km^k = k((d - 1)/2)^k$. It follows that $C_{d,k} \geq k[d/2]^k$ for any $d \geq 4$ and $k \geq 3$. □

**Proof of Assertion 1.** Let $H$ be the cyclic group $Z_m$ and let $t = p = 2$. Let $X = \{(g, 0; 1), (h, h + 1; 0) \mid g, h \in Z_m, 1 \leq h \leq m - 1, \text{where } h \text{ is odd}\}$. If $m$ is even, the Cayley digraph $C(G, X)$ is of degree $d = |X| = 3m/2$ and order $|G| = 2m^2 = 8d^2/9$, where $d$ is a multiple of 3. If $m$ is odd, the Cayley digraph $C(G, X)$ is of degree $d = (3m - 1)/2$ and order $|G| = 2(2d + 1)^2/9$, where $d \equiv 1 \pmod{3}$, $d \geq 4$.

Let us show that any element $(x_1, x_2; 1)$, where $x_1, x_2 \in Z_m, x_2 \neq 0$, can be expressed as a product of two elements of $X$. We have

$$(x_1, h; 1) = (x_1 + (h + 1)^{-1}, 0; 1)(h, h + 1; 0)$$

and

$$(x_1, h + 1; 1) = (h, h + 1; 0)(h^{-1} + x_1, 0; 1).$$

Since $(x_1, 0; 1) \in X$ and $(x_1, x_2; 0) = (x_1, 0; 1)(x_2, 0; 1)$ for any $x_1, x_2 \in Z_m$, the diameter of $C(G, X)$ is equal to 2.
A slight modification of the generating set yields analogous result for \( d \equiv 2 \pmod{3} \). Suppose that \( m \) is odd. By adjoining a new element of \( G \) to \( X \), we have a Cayley digraph of degree \( d = (3m + 1)/2 \) and order \( |G| = 2(2d - 1)^2/9 \), where \( d \equiv 2 \pmod{3}, d \geq 5 \). □

**Proof of Theorem 2.** (i) Let \( k \geq 4 \) be even, \( t = k - 1 \) and \( p = 2(k - 1) \). Let \( a_g = (g, t, \ldots, t; 1), b_h = (t, h, t, \ldots, t; k + 1) \), \( a_g, b_h \in G \) and let \( X = \{a_g, b_h \mid \) for all \( g, h \in H \}. \) The Cayley digraph \( C(G, X) \) is of degree \( d = |X| = 2m, m \geq 2 \) and order \( |G| = 2(k - 1)m^{k-1} = 2(k - 1)(d/2)^{k-1} \). In order to prove that the diameter of \( C(G, X) \) is at most \( k \), it suffices to show that we can express each element of \( G \) as a product of at most \( k \) elements of \( X \).

Let us show that any element \((x_1, x_2, \ldots, x_{k-1}; y)\), where \( x_1, x_2, \ldots, x_{k-1} \in H \) and \( y \in Z_{2(k-1)}, y \) is odd, can be obtained as a product of \( k - 1 \) elements of \( X \). We have

\[
(x_1, x_2, \ldots, x_{k-1}; r) = \left( \prod_{i=1}^{(r+1)/2} b_{x_{2i}} \right) \left( \prod_{j=1}^{k-2-r} a_{x_{j+r+1}} \right) a_{x_1} \left( \prod_{\ell=1}^{(r-1)/2} b_{x_{2\ell+1}} \right)
\]

for \( r = 1, 3, \ldots, k - 3 \), and

\[
(x_1, x_2, \ldots, x_{k-1}; k - 2 + r) = a_{x_1} \left( \prod_{i=1}^{(r-1)/2} b_{x_{2i+1}} \right) \left( \prod_{j=1}^{k-1-r} a_{x_{j+r}} \right) \left( \prod_{\ell=1}^{(r-1)/2} b_{x_{2\ell}} \right)
\]

for \( r = 1, 3, \ldots, k - 1 \).

Then any element \((x_1, x_2, \ldots, x_{k-1}; y + 1)\) can be expressed as follows:

\[
(x_1, x_2, \ldots, x_{k-1}; y + 1) = (x_1, x_2, \ldots, x_{k-1}; y)a_t.
\]

None of the elements \((x_1, x_2, \ldots, x_{k-1}; y)\), where all \( x_i \neq t, 1 \leq i \leq k - 1 \), can be obtained as a product of fewer than \( k - 1 \) elements of \( X \), therefore none of the elements \((x_1, x_2, \ldots, x_{k-1}; y + 1)\) can be obtained as a product of fewer than \( k \) elements of \( X \). It follows that the diameter of \( C(G, X) \) is equal to \( k \).

Since the last coordinate of any element in the generating set \( X \) is odd, no two different vertices \((x_1, x_2, \ldots, x_{k-1}; r)\) and \((x_1', x_2', \ldots, x_{k-1}'; r')\) of \( C(G, X) \) are adjacent if either both \( r, r' \) are even or both \( r, r' \) are odd \((r, r' \in Z_{2(k-1)}\) and \(x_i, x'_i \in H, 1 \leq i \leq k - 1\). The Cayley digraph \( C(G, X) \) is bipartite.

By adjoining to the generating set \( X \) a new element of \( G \) with an odd last coordinate, we obtain a bipartite Cayley digraph of degree \( d = 2m + 1, \)
diameter at most $k$ and order $|G| = 2(k - 1)m^{k-1} = 2(k - 1)((d - 1)/2)^{k-1}$. Hence, $BC_{d,k} \geq 2(k - 1)[d/2]^{k-1}$ for any $d \geq 4$ and any even $k \geq 4$.

(ii) Let $k \geq 5$ be odd and $t = p = k - 1$. For any $g, h \in H$ let $a_g = (g, t, \ldots, t; 1)$, $b_h = (t, t, h, t, \ldots, t; 3)$ and let $X = \{(a_g, b_h \mid g, h \in H\}$. The Cayley digraph $C(G, X)$ is of degree $d = |X| = 2m$ and order $|G| = (k - 1)m^{k-1} = (k - 1)(d/2)^{k-1}$.

We show that any element $(x_1, x_2, \ldots, x_{k-1}; y)$, where $x_1, x_2, \ldots, x_{k-1} \in H$ and $y \in \mathbb{Z}_{k-1}$, $y$ is even, can be expressed as a product of $k - 1$ elements of $X$. Let $0 \leq y \leq k - 3$. Then

$$(x_1, x_2, \ldots, x_{k-1}; y) = a_{x_1}a_{x_2}\left(\prod_{i=1}^{y/4} b_{x_{4i+1}}a_{x_{4i+2}}\right)\left(\prod_{j=1}^{k-3-y} a_{x_{j+y+2}}\right)\left(\prod_{\ell=1}^{y/4} b_{x_{4\ell-1}}a_{x_{4\ell}}\right)$$

if $y$ is a multiple of 4, and

$$(x_1, x_2, \ldots, x_{k-1}; y) = \left(\prod_{i=1}^{(y+2)/4} b_{x_{4i-1}}a_{x_{4i}}\right)\left(\prod_{j=1}^{k-3-y} a_{x_{j+y+2}}\right)a_{x_1}a_{x_2}\left(\prod_{\ell=1}^{(y-2)/4} b_{x_{4\ell+1}}a_{x_{4\ell+2}}\right)$$

if $y \equiv 2 \pmod{4}$. Similarly as in part (i) of this proof it can be checked that the Cayley digraph $C(G, X)$ is bipartite of diameter $k$.

By adding to $X$ a new element of $G$ with an odd last coordinate, we obtain a bipartite Cayley digraph of degree $d = 2m + 1$ and order $(k - 1)((d - 1)/2)^{k-1}$. Thus, $BC_{d,k} \geq (k - 1)[d/2]^{k-1}$ for any $d \geq 4$ and any odd $k \geq 5$.

(iii) Let $t = p = 2$. Let $X = \{(g, t; 1) \mid g \in H\}$. The Cayley digraph $C(G, X)$ is of degree $d = |X| = m$ and order $|G| = 2d^2$. Since for any $x_1, x_2 \in H$, we have $(x_1, x_2, 0) = (x_1, t, 1)(x_2, t, 1)$ and $(x_1, x_2, 1) = (x_1, t, 1)(x_2, t, 1)(t, t, 1)$, the diameter of $C(G, X)$ is equal to 3. If the last coordinates of any two vertices in $C(G, X)$ are equal, the vertices are non-adjacent and the digraph is bipartite. The proof is complete. \qed

References


